1. Let $\mathbf{X} \sim \mathrm{Mult}_k(n, \mathbf{p}(\theta))$, where θ is a real-valued parameter and the $p_i(\theta)$, $i = 1, \ldots, k$ are continuously differentiable functions.

(a) Derive an explicit expression for the score test statistic of H_0: $\theta = \theta_0$.

(b) What is the large sample distribution of the statistic in (a) when $\theta = \theta_0 + c/n^{\frac{1}{2}}$?

(c) In genetical linkage experiments the null hypothesis is $\theta = \frac{1}{2}$. The following are multinomial cell probabilities from three different linkage experiments:

(i) $(\frac{1}{2}\,\theta, \frac{1}{2}\,(1-\theta), \frac{1}{2}\,\theta, \frac{1}{2}\,(1-\theta))$

(ii) $((1-\theta)^2/2, 2\theta(1-\theta), \theta^2/2, (\theta^2 + (1-\theta)^2)/2)$

(iii) $(2\theta(1-\theta), \theta^2 + (1-\theta)^2)$.

Which of these experiments is best for testing $\theta = \frac{1}{2}$?

2. Let $\mathbf{X} \sim \mathrm{N}_2(\boldsymbol{\theta}, \mathbf{I})$, $\boldsymbol{\theta} \in \mathbf{R}^2$. Suppose that we are interested in testing

$$H_0 : |\boldsymbol{\theta}|^2 = \theta_1^2 + \theta_2^2 = a^2,$$

where a is a known number, against

$$H_1 : |\boldsymbol{\theta}|^2 \neq a^2$$

based on iid observations $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of $\mathbf{X}$.

(a) Find the mle of $\boldsymbol{\theta}$ under H_0.
Hint: Write $\boldsymbol{\theta}$ in polar coordinates.

3. The following are some testing situations. In each of these cases, the alternative hypothesis is the complement of the null hypothesis (in the usual parameter space). For each situation, decide which test you would use. Explain your choices. You need not describe the exact form of the test.

(a) $X_1, \ldots, X_n$ are iid $\mathrm{Exp}(\mu)$, $Y_1, \ldots, Y_m$ are iid $\mathrm{Exp}(\lambda)$. H_0: $\lambda = \mu$.

(b) $(n_{11}, \ldots, n_{rs}) \sim \mathrm{Mult}(n, p_{11}, \ldots, p_{rs})$ where $n = \sum_{i,j} n_{ij}$. H_0: $p_{ij} = r_i c_j$ where $r_i = \sum_{j=1}^{s} p_{ij}$ and $c_j = \sum_{i=1}^{r} p_{ij}$.

(c) $X_1, \ldots, X_n$ are iid $\mathrm{N}(\mu, \sigma^2)$. H_0: $\mu = \sigma^2$.

4. (a) Let $U_{(n_1)}, \ldots, U_{(n_k)}$ be the respective maxima of k independent samples of sizes $n_1, \ldots, n_k$ from U(0,1). Show that $-2 \log \prod_{i=1}^{k} U_{(n_i)}^{n_i} \sim \chi^2(2k)$.

(b) Let $V_{(n)} = \max_{1 \le i \le k} U_{(n_i)}$, where $n = \sum_1^k n_i$. Show that $V_{(n)}$ and $\prod_{i=1}^{k} U_{(n_i)}^{n_i} / V_{(n)}^{n}$ are independently distributed.

(c) Suppose that X_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, are independent with density of the form $f(x_{ij}; \theta_i) = g(x_{ij})/h(\theta_i)$, $0 \le x_{ij} \le \theta_i$. Find the LR statistic for testing H_0: $\theta_1 = \theta_2 = \cdots = \theta_k$ against the general alternative $\Theta \backslash \Theta_0$. Show that the log likelihood ratio statistic $\lambda \sim \chi^2(2(k-1))$.
Hint: Show that $F(x_{ij}; \theta_i) = h(x_{ij})/h(\theta_i)$.
Remark: This is an exact distribution. Notice that $\dim(\Theta)$-$\dim(\Theta_0)$=$k-1$, not $2(k-1)$