STAT 582 Homework 4 Due Jan. 31

1. Let $\mathbf{X} \sim N_2(\boldsymbol{\theta}, \mathbf{I})$, $\boldsymbol{\theta} \in \mathbf{R}^2$. Suppose that we are interested in testing

$$H_0 : |\boldsymbol{\theta}|^2 = \theta_1^2 + \theta_2^2 = a^2,$$

where a is a known number, against

$$H_1 : |\boldsymbol{\theta}|^2 \neq a^2$$

based on iid observations $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of $\mathbf{X}$. Derive the likelihood ratio test, or any of its asymptotically equivalent tests, of H_0 against H_1.

2. Let $X_1, \ldots, X_n$ be iid $\mathrm{U}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$.

(a) Show that $T = (U, V) = (\min X_i, \max X_i)$ is sufficient, and that $S = V - U$ is ancillary, in the sense of having a distribution which does not depend on θ.

(b) Show that the conditional distribution of T given $S = s$ is uniform on the set $\{(u, v): v - u = s, \theta - \frac{1}{2} < u < \theta + \frac{1}{2} - s\}$.

(c) Show that a 95% conditional confidence interval for θ given $S = s$ is $\frac{1}{2}(u + v) \pm .475(1 - s)$.

3. In order to assess the precision of a new measuring device, pairs of measurements are taken on a variety of objects. The measurement errors are assumed to be iid $N(0,\sigma^2)$, so that the measurements $X_{ij} \sim N(\xi_i, \sigma^2)$, $j = 1, 2$, $i = 1, \ldots, n$.

(a) Show that the mle of σ^2 converges in probability to $\sigma^2/2$.

(b) Let $\hat{\xi}_i$ be the mle of ξ_i. Are the $\hat{\xi}_i$ ancillary for σ^2?

(c) Find the conditional distribution of X_{ij} given $\hat{\xi}_i$, and show that the conditional mle of σ^2 is consistent.

4. A scientific journal, in an attempt to maintain experimental standards, insists that all reported statistical results have (classical) error probability of α_0 (or better). To consider a very simplistic model of this situation, assume that all statistical tests conducted are of the form H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$, where θ_0 is the standard, and θ_1 the new proposal. Let β denote the power of the test used, and assume that $\alpha = \alpha_0$ and β are fixed. Let π_0 be the proportion of all experiments conducted in which θ_0 is correct, and π_1 the proportion in which θ_1 is correct.

(a) Show that the proportion of articles published in the journal that have correct results is $\pi_1 \beta / (\alpha_0 + \pi_1(\beta - \alpha_0))$.

(b) The journal policy is intended to guarantee a proportion $1 - \alpha_0$ of correct articles. Under what circumstances does this happen?

(c) Show that the proportion of correct published results is never less than π_1. (You may—safely—assume that $\beta \geq \alpha_0$.)

(d) The editors of the journal decide to investigate the problem. They survey all of the recent contributions to the journal, and determine the total number , N, of experiments that were conducted by the contributors. Let x denote the number of such experiments that were reported in the journal. Assume that all tests had fixed level $\alpha_0 = 0.05$ and power $\beta = 0.8$. If $N = 500$ and $x = 75$, how would you estimate the proportion of correct articles in the journal? How good is your estimate?