1. Suppose that measurements come from either of two instruments, with observations (A_i, X_i), $i = 1, \ldots, n$, where $A_i = 1$(instrument 2 used)~Bern($\frac{1}{2}$), and $(X_i|A_i)$~N$(\theta, \sigma^2_{A_i})$.

(a) Find the mle of θ.

(b) What is the Fisher information about θ in the sample?

(c) Show that $A = \sum A_i$ is ancillary.

(d) Show that the conditional variance of $\hat{\theta}$, given $A = a$, is the reciprocal of the observed information, or $(-l''_n(\hat{\theta}))^{-1}$.

2. If the asymptotic distribution of the likelihood ratio statistic were exact, we would have $\mathbf{E}\lambda = r - s = dim\Theta - dim\Theta_0$. It is often possible to find an expansion

$$\mathbf{E}\lambda = (r-s)\left(1 + \frac{c}{n} + o(\frac{1}{n})\right)$$

The factor $1 + c/n$, where c may depend on parameters and therefore may need to be estimated, is called a **Bartlett correction**. Improved distributional properties can often be obtained by replacing λ by

$$\lambda' = \frac{\lambda}{1 + \frac{c}{n}}$$

(a) Let $X_1, \ldots, X_n$ be iid exponential (θ). Show that the Bartlett correction for testing $H_0\colon \theta = \theta_0$ is $1 + 1/(6n)$.

(b) Compare confidence limits for θ based on λ, λ', and $\hat{\theta}$ for $n = 20$. (You are not likely to be able to do this analytically.)

3. Consider testing $H_0\colon \theta = \theta_0$ vs. $H_1\colon \theta = \theta_1$ based on data $\mathbf{x}$.

(a) Show that the test rejecting H_0 if $f(\mathbf{x};\theta_1)/f(\mathbf{x};\theta_0) \geq c$ satisfies $\pi_0(c)/\pi_1(c) \leq 1/c$ (and hence $\pi_0(c) \leq 1/c$) where $\pi_i(c)$ is the power at θ_i for critical value c.

(b) Show that a 5%-level Neyman-Pearson test rejects H_0 more often when it is true than does a pure likelihood test which rejects when $c = 20$.

4. Suppose that $X_1, \ldots, X_n$ are iid random variables with density

$$f(x;\mu,\kappa) = \frac{1}{2\pi I_0(\kappa)} \exp(\kappa\mu^T x)$$

where μ and x are unit vectors in $\mathbf{R}^2$. Here I_0 is the appropriate normalizing constant, which happens to be a modified Bessel function of the first kind and order zero (don't worry: it is only a name).

(a) Show that the mle $\hat{\mu} = \sum x_i/r$ where $r = \|\sum x_i\|$ is the resultant.

(b) Show that the distribution of r depends on κ only (so that if κ is known, r is ancillary).
Hint: $\int_0^{2\pi} \exp(\kappa(\mu_1 \cos\phi + \mu_2 \sin\phi)d\phi = 2\pi I_0(\kappa)$.

(c) Compute the Barndorff-Nielsen formula for the density of the mle (do not worry about whether or not it applies here) for the case where κ is known.

(d) Show that this is the exact conditional distribution for $\hat{\mu}$, given r.