1. Let X_i be iid $\mathrm{N}(\mu, \sigma^2)$. Show that $(\bar{X}, m_2)$ is independent of $g_1 = m_3/m_2^{3/2}$, where $m_k = \sum (X_i - \bar{X})^k / n$.

2. Trials are carried out on n pairs of subjects. Each subject may give a response 0 or 1, and the responses of all subjects are independent. In each pair, one subject is a "Control", yielding response X_j, and the other "Experimental", yielding response Y_j, $j = 1, \ldots, n$. It is believed that for each j

$$\frac{\mathbf{P}(Y_j = 1)}{\mathbf{P}(Y_j = 0)} = \alpha \, \frac{\mathbf{P}(X_j = 1)}{\mathbf{P}(X_j = 0)} \quad \text{where } 0 < \alpha < \infty.$$

(a) Show that we may write

$$\mathbf{P}(Y_j = y, X_j = x) = \frac{(\alpha \beta_j)^y}{1 + \alpha \beta_j} \, \frac{\beta_j^x}{1 + \beta_j}$$

for some (unknown) $\beta_1, \ldots, \beta_n$.

(b) Observations x_j, y_j, $j = 1, \ldots, n$, are obtained. Derive the likelihood equations for the joint estimation of α with $\beta_1, \ldots, \beta_n$ and comment on their solution.

(c) Compare the approach in (b) with that of maximum likelihood estimation of α, conditional on the values of $X_j + Y_j$, $j = 1, \ldots, n$.

(d) Show that under the conditioning in (c) $\hat{\alpha}^c = S/(N - S)$, where $N = \#\{j\colon X_j + Y_j = 1\}$ and $S = \sum_{X_j + Y_j = 1} Y_j$. Demonstrate that $\hat{\alpha}^c$ is consistent if and only if $\beta_1, \ldots, \beta_n, \ldots$ are such that

$$\mathbf{P}(X_j + Y_j = 1 \ \text{ i.o.}) = 1.$$