1. Let Y_i be independent exponential random variables with means μ_i. Consider the general linear model $g(\mu_i) = \beta_0 + \beta_1 x_i$, where the β_k are unknown parameters, and x_i are observed covariates.

(a) Assuming that the link function is $g(x) = 1/x$, find the natural sufficient statistics for (β_0, β_1).

(b) Regarding β_0 as a nuisance parameter, determine the Barndorff-Nielsen formula for the appropriate conditional likelihood of β_1.

2. Suppose that $X_n = \mu + \rho(X_{n-1} - \mu) + \varepsilon_n$, where the ε_i are iid $\mathrm{N}(0, \sigma^2)$ and $|\rho| < 1$. This is an **autoregressive model of order 1**. Find a minimal sufficient statistic for (μ, ρ, σ^2)

(a) if $X_0 = \mu$;

(b) if $X_0 = X_n$ a.s.

3. Suppose that an urn contains N balls, of which an unknown number W are white and the rest are black. Suppose also that the prior distribution of W is hypergeometric with parameters A,B, and N, where A and B are positive integers with $A + B \geq N$.

(a) If you know that $r \leq W \leq s$, where r and s are integers such that $0 < r \leq s < N$, show that there are unique values of A and B such that under the prior distribution

$$\mathbf{P}\{r \leq W \leq s\} = 1, \ \ \mathbf{P}\{W = r\} > 0 \ \text{ and } \ \mathbf{P}\{W = s\} > 0.$$

(b) Suppose instead that you only know that $0 \leq W \leq N$. Show that any prior hypergeometric distribution with $A \geq N$ and $B \geq N$ will assign positive mass to each integer $0, 1, \ldots, N$.

(c) Suppose that one ball is selected from the urn at random. What is the probability under the prior distribution that it will be white?

(d) Suppose that n balls $(1 \leq n \leq N)$ are selected at random from the urn without replacement, and that w of these balls are white. Show that the posterior dist ribution of the number of white balls among those remaining in the box is hypergeometric with parameters $A - w$, $B - n + w$ and $N - n$.

4. let W be a one-dimensional parameter, and let the interval $L(x) \leq W \leq R(x)$ have posterior probability $1 - \alpha$. This interval is a *high posterior density* (HPD) interval for W if there is a constant c such that $L(x) \leq W \leq R(x)$ iff $f(w|x) \geq c$. Show that for each x, the HPD interval has shorter length than any other credibility interval (*i.e.,* than any other interval with posterior probability $1 - \alpha$).

Hint: Minimize $\displaystyle\int_{L(x)}^{R(x)} 1 dw$ subject to the constraint $\displaystyle\int_{L(x)}^{R(x)} f(w|x) dw = 1 - \alpha$.