1. (a) Let $(X|\sigma^2) \sim N(0, \sigma^2)$. Find a natural conjugate prior for σ^2, and show that it is the distribution of $1/Y$ where Y has a gamma distribution.

(b) Let $(X|\mu, \kappa) \sim N(\mu, 1/\kappa)$, both parameters unknown. Assume that the priors for μ and κ are independent, both taken to have the natural conjugate form for when the other parameter is known. Show that this yields a conjugate prior distribution.

(c) Find the marginal posterior distribution for μ in (b).

2. On visiting a new cafeteria, a distinguished statistician took five cubes of sugar for his coffee. On each wrapper was a picture of a bird. Out of the first four, the third was a cardinal, but the other three were swallows. What bird is likely to appear on the fifth wrapper?

3. Let the sample space be {0,1}, and the parameter space {1,2}. Assume a loss function is given by $L(1,1) = L(2,2) = 0$ and $L(1,2) = a$, $L(2,1) = b$. Suppose that we may observe *either* X or Y where

$$\mathbf{P}_1(X = 1) = 2/3 \quad \mathbf{P}_2(X = 1) = \tfrac{1}{2}$$
$$\mathbf{P}_1(Y = 1) = 3/4 \quad \mathbf{P}_2(Y = 1) = \tfrac{1}{2}$$

Let $\boldsymbol{\lambda} = (\lambda, 1 - \lambda)$ be the prior distribution.

(a) Find the Bayes risk when X is observed, and when Y is.

(b) In the case $a = b$, $\boldsymbol{\lambda} = (\frac{1}{2}, \frac{1}{2})$, would it be better to observe X or Y? Find the Bayes rules and corresponding Bayes risks.

(c) For the general case $a \neq b$, $\lambda \neq \frac{1}{2}$, would it be better to observe X or Y?

4. Suppose that $\Omega = \{w_0, w_1\}$ with prior probabilities π_0 and π_1, and that $\mathbf{X}$ is a discrete random vector with likelihood function $L(w)$. Let $\lambda(\mathbf{X})$ be the Neyman-Pearson test statistic for testing $W = w_0$ against $W = w_1$.

(a) Show that $\mathbf{P}(W = w_0|\mathbf{X}) = (1 + \pi_1 \lambda(\mathbf{X})/\pi_0)^{-1}$.

(b) A *Bayes test* rejects the null hypothesis when its posterior probability is at most α. Show that the Bayes test of $W = w_0$ rejects for $\lambda(\mathbf{X}) > c$, and determine the rejection constant c.