1. Let X be 0-1-valued with

$$\mathbf{E}(X|p) = p + .5 \times 1(p = 1/4) - .5 \times 1(p = 3/4)$$

where $0 \le p \le 1$. Assume that p is uniformly distributed *a priori*. Show that the posterior mean, given n iid observations of X is not a consistent estimator for $p = 1/4$ or $p = 3/4$. How about for other values of p?

2. The Poisson distrbution is a good model for the variability of the number of α-particles that enter a detector, emitted by a source containing polonium. The average number of particles that enter the detector per second, during a period of observation that is short compared to the half-life of polonium (140 days), is a constant $\lambda > 0$. Not every α-particle that enters the detector is actually detected, and the *efficiency* of the deterctor is the probability E of actually detecting an α-particle that enters the detector. The random variable M counts the number of α-particles that enter the detector in a specific interval of length 1 second, and the random variable N counts how many of them are actually detected.

(a) Given that $N = n$ α-particles were detected during the 1 second period under consideration, and conditionally upon $E = e$, find the probability distribution of the number of α-particles that entered the detector but were undetected.

(b) Given that $N = n$ and $E = e$, find the estimator of M with smallest mean squared error.

(c) Compute a Bayes estimator of E with respect to squared error loss.

3. Consider the normalized likelihood $\phi(w|x) = L_x(w)/\int L_x(w)dw$. Suppose the W has prior $f(w)$ and posterior $f(w|x)$. For any set A let α be such that $\int_A \phi(w|x)dw \ge 1 - \alpha$. Set

$$\beta = \frac{\sup_A f(w)}{\inf_A f(w)} - 1 \qquad \gamma = \max(0, \frac{\sup_{A^c} f(w)}{\inf_A f(w)} - 1).$$

Show that

$$\int |\{\phi(w|x) - \xi(w|x)\}|dw \le \varepsilon,$$

where

$$\varepsilon = \max(\frac{\alpha+\beta}{1-\alpha}, \frac{\alpha+\beta+\alpha\gamma}{1+\alpha+\beta+\alpha\gamma}) + \frac{\alpha(2-\alpha+\gamma)}{1-\alpha}$$

Hint: Show that $f(w|x)/\phi(w|x) = f(w)/\int f(w)\phi(w|x)dw$, and bound the denominator of the right-hand side above and below.

4. How sharp is the inequality in 3? For $\alpha = 0.01$, $\beta = 0.01$, and $\gamma = 10$, find ϕ, f, and A, to make the left-hand side of the inequality as large as you can, and calculate the ratio of the left-hand side to ε.