STAT 583 Homework 2 Due April 10

This homework must be handed in at the beginning of class.

1. Consider a Bayesian testing situation with $H_0\colon W = 0$ and $H_1\colon W \neq 0$, where $(X_1, \ldots, X_n | W = w) \overset{\text{iid}}{\sim} N(w, 1)$. Given that $W \neq 0$ we assume a $N(\mu, 1/\tau)$ prior. Define the **Bayes factor**, or the odds[1)] for H_0 from the experiment of observing $x = \sum_i x_i/n$ as the ratio $R(x) = p_0(x)/p_1(x)$ of the predictive densities under H_0 and H_1, respectively.

(a) Show that the posterior odds in favor of H_0 can be written as the Bayes factor times the prior odds.

(b) Show that the Bayes factor is

$$R(x) = \left(\frac{\tau + n}{\tau}\right)^{\frac{1}{2}} \exp\left(\frac{n}{2}\left(\frac{\tau}{\tau + n}(\mu - x)^2 - x^2\right)\right)$$

(c) Suppose that the prior odds is 1 (i.e., that the odds are even...), that $\tau = 1$ and $\mu = 0$. Find the indifference point $x*$ for which $R(x*) = 1$, and show that it decreases with n.

(d) For the indifference point $x*$ from (c), let $z = \sqrt{n}|x*|$. For what values of n is $z > 1.96$?.

(e) Theory predicts that W, the melting point of a particular substance under a pressure of 10^6 atm is 4.01°C. Due to the high pressure, the determination of the melting point is somewhat uncertain. Indeed, the measurement error can be thougt of as standard normal. Five independent experiments yield 4.9, 4.3, 5.1, 4.6 and 2.9. Assuming that the prior probability that $W = 4.01$ is 0.8, while all other values of W have density 0.2×N(4.01,1/4). Conduct a Bayes test of the theoretical prediction.

2. Let X have a geometric distribution with

$$\mathbf{P}(X = x) = p(1 - p)^x, \quad x = 0, 1, \ldots, \quad p \in (0, 1).$$

(a) What is the natural parameter for this family? Find the natural conjugate prior, and verify that when transformed to a prior on the parameter p it is a beta distribution.

(b) Find the Laplace approximation to the posterior mean of the population mean $\mu = p/(1 - p)$ after observing n geometric variates, and using a beta(α, β)-prior for p.

(c) For $\alpha = 1$, $\beta = 2$, $n = 53$, and $\sum x_i = 77$, compare the approximation in (b) to the true value.

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1) odds = probability / (1 - probability)