STAT 583 Homework 4 Due April 24

1. Let $T(F) = F^{-1}(\frac{1}{2})$. Assume that $f(F^{-1}(\frac{1}{2}) > 0$ where $f = F'$.

(a) Show that $F^{-1}(\frac{1}{2})$ is unique.

(b) Show that if $F_\lambda = F + \lambda(G - F)$ we have

$$(1-\lambda)F(F_\lambda^{-1}(\tfrac{1}{2})) + \lambda G(F_\lambda^{-1}(\tfrac{1}{2}))) = \tfrac{1}{2}$$

Hint: Recall that $F_\lambda^{-1}(\frac{1}{2}) = \inf_x\{F(x) + \lambda(G(x) - F(x)) \ge \frac{1}{2}\}$.

(c) Show that

$$d_1T(F; G - F) = -\frac{G(F^{-1}(\frac{1}{2})) - \frac{1}{2}}{f(F^{-1}(\frac{1}{2}))}.$$

Hint: Implicit differentiation of the relation in (b).

(d)) Show that T does not have a differential.
Hint: Let $F(x) = x, 0 \le x \le 1$ and

$$G(x) = \begin{cases} x + \varepsilon & 0 \le x < \frac{1}{2} - \varepsilon \\ \frac{1}{2} & \frac{1}{2} - \varepsilon \le x \le \frac{1}{2}. \\ x & \frac{1}{2} < x \le 1 \end{cases}$$

2. (a) Show that for a functional $T(F)$ with linear derivative d_1 and expansion

$$T(F_n) = T(F) + d_1T(F; F_n - F) + R_{1n}$$

the condition $R_{1n} \xrightarrow{\mathbf{P}} 0$ implies that

$$T(F_n) \xrightarrow{\mathbf{P}} T(F) + \mathbf{E}_F h(F; X)$$

(so that the second term on the right-hand side is the asymptotic bias of the estimator $T(F_n)$) where $h(F; x) = d_1T(F; \delta_x - F)$ and δ_x is point mass at x..

(b) Apply this result to $T(F) = (\mathbf{E}_F(X))^2$.

3. Let F be a cdf on [0,1], and define

$$T(F) = \sum_{x\in[0,1]} (F(x) - F(x-))^\alpha$$

where $\alpha > 1$.

(a) Find the derivative of T at the uniform distribution on [0,1].

(b) Show that

$$n^{\frac{1}{2}}R_{1n} = n^{\frac{1}{2}}(T(F_n) - T(U) - d_1T(U; F_n - U)) \to 0 \text{ in probability if } \alpha < 3/2.$$

4. Let $X_1, \ldots, X_n$ be iid nonnegative random variables with cdf $F(x) = (1/\mu)\int_0^x t\,dG(t)$, where G is a cdf on $[0,\infty)$ with mean $0 < \mu = \int_0^\infty t\,dG(t) < \infty$.

(a) Show that $\mathbf{E}(1/X) = 1/\mu$.

(b) Show that $G(x) = \int_0^x (\mu/y) dF(y)$.

(c) Use the results of (a) and (b) to suggest an estimator of $G(x)$ based on the X's. Outline how you would determine the asymptotic properties of the estimator.