1. Let $T(F) = \mu_F^2 = (\int x dF(x))^2$. Find, using the asymptotic theory for statistical functionals, the limiting distribution of $T(F_n)$ when $\mu_F \neq 0$.

2. Find and sketch the influence function $d_1 T(F; \delta_x - F)$ as a function of x for the following functionals:

(a) $T(F) = \int x dF(x)$.

(b) $T(F) = F^{-1}(p)$, $0 < p < 1$, assuming F differentiable at $F^{-1}(p)$.

(c) $T(F) = \sum_{i=1}^{k} (\int_{A_i} dF(x) - p_i)^2 / p_i$, where $p_i = \int_{A_i} dF_0(x)$ and the A_i form a partition of the real line.

(d) $T(F) = \frac{1}{1-2\alpha} \int\limits_{\alpha}^{1-\alpha} F^{-1}(p) dp$

Hint: Use (b). You need to consider separately $x < F^{-1}(\alpha)$ and $x > F^{-1}(1-\alpha)$.

3. Suppose that $T(G) = \int \cdots \int h(x_1, \ldots, x_k) \prod_{i=1}^{k} d(G(x_i) - F(x_i)$ for a fixed cdf F.

(a) Show that $T(G)$ can be written in the form

$$\int \cdots \int \tilde{h}(x_1, \ldots, x_k) \prod_1^k dG(x_i),$$

where $\tilde{h}$ depends on F.
Hint: Start with $k = 1$ and 2.

(b) Show that $\int \tilde{h}(x_1, \ldots, x_k) dF(x_i) = 0$, $1 \leq i \leq k$.

(c) Deduce that if d_k is k-linear we have

$$d_k T(F; F_n - F) = n^{-k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \tilde{T}_k(F; x_1, \ldots, x_k)$$

for some appropriately chosen $\tilde{T}_k$.

4. Let $F_n^{(-j)}$ be the edf of $\{X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}$. Define a **pseudo-value** for a statistical functional $T(F)$ by

$$T_j = nT(F_n) - (n-1)T(F_n^{(-j)})$$

An estimate of $\mathbf{Var}_F(T(F_n))$ is $\hat{\sigma}_n^2 = n^{-2} \sum_1^n (T_j - \bar{T})^2$. This is called the **jackknife** estimate of variance.

(a) Suppose that $T(F_n) = n^{-1} \sum_1^n \psi(X_j)$. Show that $n\hat{\sigma}_n^2 \to \mathbf{Var}_F \psi(X)$ in probability.

(b) Let $T(F) = \frac{1}{2}(\inf \{x : F(x) \geq \frac{1}{2}\} + \sup \{x : F(x) \leq \frac{1}{2}\})$ be the midpoint of the interval of medians of F. Find the jackknife estimate of the variance of $n^{\frac{1}{2}} T(F_n)$, the midpoint of the interval of medians of F. Find the jackknife estimate of the variance of $n^{\frac{1}{2}} T(F_n)$, and show that when $n = 2k$ and F is the uniform distribution on [0,1], then $\hat{\sigma}_n^2$ converges in distribution to $Y^2/4$ where

$Y \sim \mathrm{Exp}(1)$ (rather than to the right value 1/4).
Hint: You may use that $n(X_{(k)} - X_{(k-1)}) \xrightarrow{d} \mathrm{Exp}(1),\ k = 1, \ldots, n-1$ where $X_{(i)}$ is the ith order statistic.