1. Let F be the cdf of a positive rv X with finite mean $\mu = \mathbf{E}_F(X) = \int_0^\infty (1 - F(x))dx$ The **mean residual life** functi on is defined as

$$e(x) = \mathbf{E}(X - x|X > x).$$

(a) Show that $e(x) = \int_x^\infty (1 - F(y))dy \,/\, (1 - F(x))$ and express $T(F_n)$ in terms of $x_1, \ldots, x_n$.

(b) Fix $x_0 > 0$ and write $T(F) = e(x_0)$. Find $IF(x)$, express the asymptotic variance of $n^{\frac{1}{2}}(T(F_n) - T(F))$ in terms of the residual life variance $\mathbf{Var}(X|X > x_0)$, and determine the gross error sensitivity of T.

(c) Prove the asymptotic normality of $n^{\frac{1}{2}}(T(F_n) - T(F))$.

2. Let $T(F) = \int F^{-1}(t)dM(t)$ where $M(t)$ is a cdf on $[\alpha, 1 - \alpha]$, $0 < \alpha \leq \frac{1}{2}$, and α is taken as large as possible, and F is a cdf on $\mathbf{R}$ with positive density.

(a) Describe $T(\mathbf{F}_n)$.

(b) Show that

$$IF(x) = \int \frac{s}{f(F^{-1}(s)} \, dM(s) - \int_{F(x)}^{1} \frac{1}{f(F^{-1}(s)} \, dM(s).$$

3. The α-Winsorized mean $W = W(F_n)$ is obtained by replacing the αn leftmost observations by $x_{(\alpha n+1)}$, and the αn rightmost observations by $x_{(n(1-\alpha))}$, and taking the mean of the modified sample.

(a) Show that $W(F) = (1 - 2\alpha)T(F) + \alpha F^{-1}(\alpha+) + \alpha F^{-1}(1 - \alpha)$ where $T(F)$ is the α-trimmed mean.

(b) Find the influence curve for W, and show that it is discontinuous at $F^{-1}(\alpha)$ and $F^{-1}(1 - \alpha)$. *Hint:* Use 2(b).

(c) The heuristic idea behind Winsorizing is to avoid throwing away the extreme observations, but only reduce their influence on the estimate to that of more moderate order statistics. Argue, by looking at the influence functions, that the trimmed mean has this behavior, whereas the Winsorized mean does not.

(d) Show that the pseudo-values for the α-trimmed mean are the Winsorized sample. Deduce that the jackknife estimate of the variance of the trimmed mean is a suitably scaled Winsorized variance (you may assume that $(n - 1)\alpha$ is an integer).

4. Let $X|W = w \sim f(x|w)$. Fix x and consider the Bayes estimate of W for squared error loss with prior density $\pi(w)$:

$$T(\pi) = \frac{\int wf(x|w)\pi(w)dw}{\int f(x|w)\pi(w)dw}.$$

(a) Let $\pi(w) = (1 - \varepsilon)g(w) + \varepsilon h(w)$, and show that

$$d_1T(g;h-g) = a(x)(T(h)-T(g))$$

where $a(x) = (\int f(x|w)h(w)dw)/(\int f(x|w)g(w)dw)$.

(b) Letting h correspond to point mass at w we can measure the sensitivity of T to changes in the prior by

$$\gamma = \sup_w |d_1T(g;h-g)|,$$

holding g fixed. Show that

$$\gamma = \sup_w |(w-T(g))\frac{f(x|w)}{\int f(x|w)g(w)dw}|.$$

(c) Suppose that $X|W=w$~Bin$(2,w)$, and that g is uniform on [0,1]. For what values of x is $T(g)$ least sensitive to changes in the prior? For those x-values, where should you put prior mass in order to obtain the largest change in T?

You may use the well known fact that if X has density $B(\alpha,\beta)^{-1}x^{\alpha-1}(1-x)^{\beta-1}$, then $\mathbf{E}X = \alpha/(\beta+\alpha)$.