STAT 583 Homework 7 Due May 15

1. Let $T(F) = \mu_F^2 = (\int xdF(x))^2$. Find, using the asymptotic theory for statistical functionals, the limiting distribution of $T(F_n)$ when $\mu_F = 0$.

2. Let $T(F) = \int F^{-1}(t)dM(t)$ where $M(t)$ is a cdf on $[\alpha, 1-\alpha]$, $0 < \alpha \leq \frac{1}{2}$, and α is taken as large as possible, and F is a cdf on $\mathbf{R}$ with positive density. Show that the breakdown point of T is α.
Hint: The worst case is the same as for location M-estimators. Compute b_+ and b_-, and deduce that $\varepsilon^* \geq \alpha$. Now argue that $\varepsilon^* \leq \alpha$.

3. Let T be a location M-estimator, and define $\lambda(t, F) = \int \psi(x-t)dF(x)$.

(a) Show that if ψ is bounded and $\lambda(t, F_0)$ has a unique zero at $t = T(F_0)$ then $T(F) \to T(F_0)$ as $d_L(F, F_0) \to 0$.
Hint: Prove that $\lambda(t+\varepsilon, F_0) - \|\psi\|\varepsilon \leq \lambda(t, F) \leq \lambda(t-\varepsilon, F_0) + \|\psi\|\varepsilon$ where $\|\psi\| = \psi(\infty) - \psi(-\infty)$.

(b) Show that if ψ is not bounded, then $T(F)$ need not converge to $T(F_0)$.

(c) Show that if $T(F_0)$ is not unique, then $T(F)$ need not converge to $T(F_0)$.

4. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be observations from the unit circle, so that $\mathbf{x}_i^T \mathbf{x}_1 = 1$, and assume that they are iid with density $f(\mathbf{x}; \boldsymbol{\mu})$, where $\boldsymbol{\mu}^T \boldsymbol{\mu} = 1$. Define the **directional mean** of f as $T(F) = C(F)/\|C(F)\|$, where $C(F) = (\int ydF_1(y), \int ydF_2(y))^T$ and F_1, F_2 are the marginal distributions of each coordinate of $\mathbf{x}$.

(a) Show that the influence function of T and F with $C(F) \neq 0$ is

$$IF(\mathbf{x}) = \frac{\mathbf{x}\|C(F)\|^2 - C(F)\mathbf{x}^T C(F)}{\|C(F)\|^3}$$

where $\|\cdot\|$ is the Euclidean norm.

(b) Show that $\|IF(\mathbf{x})\| = (1 - |\mathbf{x}^T T(F)|^2)^{\frac{1}{2}} / \|C(F)\|$.

(c) What is the gross error sensitivity of T? Is T B-robust? Which points have the largest influence on T?