1. Let $X_1, \ldots, X_n$ be iid $f(x;\theta)$, where $X_i \in \{1, \ldots, L\}$. Estimate $\theta$ by maximum likelihood, and let $\hat{\mathbf{f}} = (f(1,\hat\theta), .., f(L,\hat\theta))^T$. Draw $X_i^* \overset{iid}{\sim} \hat{\mathbf{f}}$, and let $p_i^* = \#(X_j^* = i)/n$, and write $\mathbf{p}^* = (p_1^*, \ldots, p_L^*)^T$.

(a) Show that

$$\hat\theta(\mathbf{p}^*) \approx \hat\theta + \boldsymbol{\eta}(\hat\theta)^T(\mathbf{p}^* - \hat{\mathbf{f}})/i_{\hat\theta},$$

where $\boldsymbol{\eta}(\hat\theta) = \dfrac{\partial}{\partial\theta}\log\hat{\mathbf{f}}$ and $i_\theta$ is the Fisher information in a single observation.

(b) Deduce that $\mathbf{Var}^*\hat\theta(\mathbf{p}^*) \approx (ni_{\hat\theta})^{-1}$.

2. Let $\theta$ be a real parameter, estimated by $\hat\theta = \theta(F_n)$ Assume that $\hat\theta - \theta$ is **pivotal** (ancillary for $\theta$), i.e., that its distribution is the same for all $F$ of interest, including $F_n$. Let $\hat\theta\,*$ be a bootstrap version of $\hat\theta$.

(a) Show that for all $z_1, z_2 > 0$

$$\mathbf{P}_F(\theta \in (\hat\theta - z_2, \hat\theta + z_1)) = \mathbf{P}_*(\hat\theta\,* \in (\hat\theta - z_1, \hat\theta + z_2)).$$

(b) Define $\hat\theta_{UP}$ and $\hat\theta_{LO}$ as upper and lower $\alpha$-quantiles of the bootstrap distribution of $\hat\theta\,*$. Show that

$$\mathbf{P}_F(\theta \in (2\hat\theta - \hat\theta_{UP}, 2\hat\theta - \hat\theta_{LO})) = \mathbf{P}_*(\hat\theta\,* \in (\hat\theta_{LO}, \hat\theta_{UP})) = 1 - 2\alpha.$$

(c) Assume now that there is a function $\phi = g(\theta)$, with $g$ increasi ng, and instead of assuming that $\hat\theta - \theta$ is pivotal, assume that $\hat\phi - \phi$ is (where $\hat\phi = g(\hat\theta)$). In addition, assume th at $\hat\phi - \phi$ is symmetrically distributed about 0. Use the result in (b) to construct a 1-2$\alpha$ confidence interval for $\theta$. *Remark:* You don't need to know $g$—just believe that it exists!

3. The following are measurements of the passage time of light, made by the Canadian astronomer Simon Newcomb (who invented the contaminated normal distribution) in 1882. The actual measurements (in $\mu$s) were the given values times $10^{-3}$ plus 24.8.

| | | | | |
|---|---|---|---|---|
| 28 | -44 | 29 | 30 | 26 |
| 27 | 22 | 23 | 33 | 16 |
| 24 | 29 | 24 | 40 | 21 |
| 31 | 34 | -2 | 25 | 19 |

(a) Compute the 0%, 10%, 25%, 50% trimmed mean, the Huber estimate, and the 10% Winsorized mean.

(b) The true value (based on current measurements) is 33.02. Comment on the performance of the estimators.