1. Modes of Convergence

1.1. Convergence in distribution

$$F_n(x) = \mathbf{P}(X_n \le x); \quad F(x) = \mathbf{P}(X \le x)$$

Definition 1.1.1. $X_n \xrightarrow{d} X$ if $F_n(x) \to F(x)$ at all continuity points of the latter.

Note: No random variables are needed, just distribution functions.

Theorem 1.1.1. Let $\phi_n(t) = \int \exp(itu) dF_n(u)$. If $\phi_n(t) \to \phi(t)$ with the limit continuous at 0, then $X_n \xrightarrow{d} X$ where the cdf F of X satisfies $\phi(t) = exp(itu)dF(u)$.

Definition 1.1.2. X_n~AsN(μ_n, σ_n^2) if $(X_n - \mu_n)/\sigma_n \xrightarrow{d}$ N(0, 1).

Theorem 1.1.3. If $X_n \xrightarrow{d} X$, where X is a continuous random variable, then the convergence of distribution functions is uniform.

Note: This is useful for computing asymptotic confidence bands.

Theorem 1.1.4. Let X_i~(μ_i, σ_i^2). If $s_n^2 = \sum_1^n \sigma_i^2$ satisfies $\max \sigma_i^2 / s_n^2 \to 0$ and $s_n^2 \to \infty$, then $s_n^{-1} \sum_{i=1}^n (X_i - \mu_i) \xrightarrow{d}$ N(0, 1) iff the *Lindeberg condition*

$$s_n^{-2} \sum_{i=1}^n \int_{|t-\mu_i| > \varepsilon s_n} (t - \mu_i)^2 dF_i(t) \to 0 \quad \text{for all } \varepsilon > 0$$

holds.

2. Convergence in probability

Definition 1.2.1. $X_n \xrightarrow{\mathbf{P}} X$ if $\mathbf{P}(|X_n - X| > \varepsilon) \to 0$ for all $\varepsilon > 0$.

Theorem 1.2.3. Let $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{\mathbf{P}} c$. Then

$$X_n + Y_n \xrightarrow{d} X + c \tag{i}$$

$$X_n Y_n \xrightarrow{d} cX \tag{ii}$$

$$X_n / Y_n \xrightarrow{d} X/c \quad \text{if } c \ne 0 \tag{iii}$$

Theorem 1.2.4. If $X_n \xrightarrow{\mathbf{P}} c$ and g is continuous at c then $g(X_n) \xrightarrow{\mathbf{P}} g(c)$.

Theorem 1.2.5. Let X_n~AsN(μ, σ_n^2) where $\sigma_n \to 0$, and let g be differentiable at μ with $g'(\mu) \ne 0$. Then $g(X_n)$~AsN$(g(\mu), [g'(\mu)]^2 \sigma_n^2)$.