1.3. Almost sure convergence

Definition 1.3.1 $X_n \stackrel{a.s.}{\rightarrow} X$ if $\mathbf{P}\{\omega\colon X_n(\omega) \nrightarrow X(\omega)\} = 0$.

Theorem 1.3.2 $X_n \stackrel{a.s.}{\rightarrow} X$ iff $\mathbf{P}(\}|X_n - X| > \varepsilon\} i.o.) = 0$.

Theorem 1.3.4 (Strong law of large numbers)
(i) Let (X_n) have identical distributions (μ, σ^2), with $\mathbf{Cov}(X_i, X_j) = 0$ for $i \neq j$. Then $\bar{X}_n \stackrel{a.s.}{\rightarrow} \mu$.
(ii) Let (X_n) be iid. Then $\bar{X}_n \stackrel{a.s.}{\rightarrow} \mu$ iff $\mu < \infty$.

1.4. The Skorokhod construction

Theorem 1.4.1 If $X_n \stackrel{d}{\rightarrow} X$, then there is a probability space with random variables $X_n^{'} \stackrel{d}{=} X_n$ and $X^{'} \stackrel{d}{=} X$ such that $X_n^{'} \stackrel{a.s.}{\rightarrow} X^{'}$.

Theorem 1.4.2 (Mann-Wald) Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}$ be $\mathbf{R}^k$-valued random variables, and $g\colon \mathbf{R}^k \rightarrow \mathbf{R}^l$ a continuous function. Then
(a) $\mathbf{X}_n \stackrel{a.s.}{\rightarrow} \mathbf{X}$ implies that $g(\mathbf{X}_n) \stackrel{a.s.}{\rightarrow} g(\mathbf{X})$.
(b) $\mathbf{X}_n \stackrel{P}{\rightarrow} \mathbf{X}$ implies that $g(\mathbf{X}_n) \stackrel{P}{\rightarrow} g(\mathbf{X})$.
(a) $\mathbf{X}_n \stackrel{d}{\rightarrow} \mathbf{X}$ implies that $g(\mathbf{X}_n) \stackrel{d}{\rightarrow} g(\mathbf{X})$.

Relationships
$X_n \stackrel{a.s.}{\rightarrow} X$ => $X_n \stackrel{P}{\rightarrow} X$ => $X_n \stackrel{d}{\rightarrow} X$
$X_n \stackrel{P}{\rightarrow} X$ => $\exists(n_k)\colon X_{n_k} \stackrel{a.s.}{\rightarrow} X$
$X_n \stackrel{d}{\rightarrow} c$ => $X_n \stackrel{P}{\rightarrow} c$
$X_n \stackrel{d}{\rightarrow} c$ => $\exists(X_n^{'}, X^{'})\colon X_n^{'} \stackrel{a.s.}{\rightarrow} X^{'}$

Theorem 1.4.4. Let $\xi_p = F^{-1}(p)$ be a continuity point of F. Let $F_n(x) = (1/n)\sum 1(X_i \leq x)$, and $\hat{\xi}_{np} = F_n^{-1}(p)$.
(Cor) If F has continuous derivative f at ξ_p, then $\hat{\xi}_{np}$~AsN$(\xi_p, p(1-p)/(nf^2(\xi_p)))$.
(iii) $\mathbf{P}(n^{\frac{1}{2}}(\hat{\xi}_{np} - \xi_p) \leq 0) = \Phi(0) = \frac{1}{2}$.