2.2 Asymptotic testing

Definition 2.2.1: A density $f(x;\theta)$ has **monotone likelihood ratio** (MLR) if there is a statistic $T(x)$ such that for all $\theta < \theta'$ we have $f(x;\theta) \not\equiv f(x;\theta')$ and $f(x;\theta')/f(x;\theta)$ is a nondecreasing function of $T(x)$.

Theorem 2.2.1: (MLR)
Let θ be a real parameter, and assume that $X \sim f(x;\theta)$ is MLR in $T(x)$.

(i) For testing $H_0\colon \theta \le \theta_0$ against $H_1\colon \theta > \theta_0$ there is a UMP test $\phi(x) = 1(T(x) > C) + \gamma 1(T(x) = C)$, where C and γ are determined by $\mathbf{E}_{\theta_0}\phi(X) = \alpha$.

(ii) $\pi(\theta) = \mathbf{E}_\theta \phi(X)$ is strictly increasing for all θ such that $\pi(\theta) < 1$.

Theorem 2.2.2: (Nonexistence of UMP tests)
Assume that $f(x;\theta)$ is continuously differentiable in θ with range independent of θ. Then

(i) there is no UMP test of $\theta = \theta_0$ against $\theta \ne \theta_0$.

(ii) the score function evaluated at θ_0 maximizes the slope of the power function at θ_0 for all tests of a given level.

Remark: The test rejecting $\theta = \theta_0$ against $\theta > \theta_0$ for large values of $S_n(\theta_0)$ is called the LMP test, or the **score test.**

Definition 2.2.2: The likelihood ratio test rejects $H_0\colon \theta = \theta_0$ in favor of $H_1\colon \theta \ne \theta_0$ if $\Lambda_n = L_n(\theta)/L_n(\hat\theta)$ is small, or equivalently if $\lambda_n = 2(l_n(\hat\theta_n) - l_n(\theta_0))$ is large.

Theorem 2.2.3: (Null distribution of LRT and related tests)
Under the assumptions of Theorem 2.1.2

(i) the LRT $\lambda_n \xrightarrow{d} \chi^2_s$ where $s = dim(\Theta)$;

(ii) the score test $R_n = \dfrac{1}{n^{\frac12}} S_n^T(\theta_0) I_1(\theta_0)^{-1} \dfrac{1}{n^{\frac12}} S_n(\theta_0) \xrightarrow{d} \chi^2_s$;

(iii) the Wald test $W_n = n^{\frac12}(\hat\theta_n - \theta_0)^T \hat I n^{\frac12}(\hat\theta_n - \theta_0) \xrightarrow{d} \chi^2_s$, where $\hat I$ is any consistent estimate of $I_1(\theta_0)$.

Theorem 2.2.5: (Local power of LRT and related tests)
Consider a sequence of local alternatives $\theta_n = \theta_0 + tn^{-\frac12}$. Under θ_n each of λ_n, W_n, and R_n converge in distribution to a noncentral χ^2-distribution with non-centrality parameter $\frac12 t^T I_1(\theta_0) t$.