5. Statistical functionals

Definition: The Gâteaux derivative of a statistical functional $T(F)$ is the limit

$$d_1T(F; G-F) = \lim_{\lambda \downarrow 0} \frac{T(F + \lambda(G-F)) - T(F)}{lammbda}$$

If $Q(\lambda) = T(F + \lambda(G-F))$ has a Taylor expansion, we get an expansion (the **von Mises expansion**) of T by noting that $Q(0) = T(F)$, $Q(1) = T(G)$, $Q_+{}'(0) = d_1T(F; G-F)$ etc., yielding

$$T(G) = T(F) + \sum_{k=1}^{m} \frac{1}{k!} d_k(T(F; G-F) + R_m(G)$$

We are usually particularly interested in $G = F_n$, and we write $R_m(F_n) \equiv R_{mn}$.

Definition: T has a **differential** if there is a linear functional $\dot{T}(F; \Delta)$ (i.e., $\dot{T}(F; \sum a_i\Delta_i) = \sum a_i \dot{T}(F; \Delta_i)$) such that for all G

$$T(G) - T(F) - \dot{T}(F; G\_F) = o(\|G-F\|)$$

for some appropriate norm $\|\cdot\|$. $\dot{T}$ is also called the **Freche/*'t** derivative of T.

Theorem 5.1: If T has a differential at F with respect to $\|\cdot\|$, then, for any G, $d_1T(F; G-F)$ exists and equals $\dot{T}(F; G-F)$.

Theorem 5.2: Let T have a differential at F with respect to $\|\cdot\|$. Let (X_i) be observations (not necessarily independent) from F, such that $n^{\frac{1}{2}}\|F_n - F\| + O_{\mathbf{P}}(1)$. Then $n^{\frac{1}{2}} R_{1n} \xrightarrow{\mathbf{P}} 0$.

Define the **influence curve** of T at F by $h(F, x) = d_1T(F; \varepsilon_x - F)$ where ε_x is the cdf of point mass at x.

Theorem 5.3: Suppose that T has a linear derivative d_1 satisfying

(i) $O < \mathbf{Var}_F h(F; X) < \infty$

(ii) $n^{\frac{1}{2}} R_{1n} \xrightarrow{\mathbf{P}} 0$.

Define $\mu(T, F) = \mathbf{E}_F h(F; X)$ and $\sigma^2(T, F) = \mathbf{Var}_F h(F; X)$. Then

$$T(F_n) \sim \mathrm{AsN}(T(F) + \mu(T, F), \sigma^2(T, F)/n).$$

Theorem 5.4: Let (X_i) be iid F. Assume that $\mathbf{Var}_F h(F; X) < \infty$, $\mathbf{Var}_F h(F; X_1, X_2) > 0$ and $nR_{2n} \xrightarrow{\mathbf{P}} 0$. Also assume that $h(F, y, x) = h(F; x, y)$ and that $\mathbf{E}_F h(F; x, X) \equiv 0$ as a function of x. Then

$$n(T(F_n) - T(F) - \mu(T, F)) \xrightarrow{d} \sum_{k=1}^{\infty} \lambda_k V_k$$

where the V_k are iid χ_1^2-rv's, and λ_k are the eigenvalues of the operator

$$Ag(x) = \int (h(F; x, y) - \mu(T, F)) g(y) dF(y)$$

corresponding to the eigenfunction $g_k(x)$.