6. Robustness

Definitions: A sequence T_n of estimators is **qualitatively robust** at $F = F_0$ if the sequence of mappings taking F into the distribution of T_n under F is equicontinuous with respect to a suitable metric.

The **influence curve** $IF(x) = d_1 T(F; \varepsilon_x - F)$ measures the influence of an additional observation at x on the estimator $T_n = T(F_n)$, given a large sample from F.

The **breakdown point** of T is the smallest fraction of gross errors that carries the estimator beyond all bounds.

The **gross error sensitivity** (GES) of T is $\sup_x |IF(x)|$.

An estimator is **B-robust** if it has bounded GES.

The **efficiency** of T at the distribution F is $I_1(F) \int IF(x)\, 2dF(x)$ where $I_1(F)$ is the Fisher information of a single observation from F.

Theorem 6.1: An estimator T is asymptotically efficient iff

$$IF(x) = I_1(\theta_0)^{-1} \frac{\partial}{\partial\theta} \log f_\theta(x)\big|_{\theta=\theta_0}$$

Definition: A **location M-estimator** is the solution to $\int \psi(x - T(F))dF(x) = 0$, where ψ is increasing.

Theorem 6.2: The breakdown point of a location M-estimator is $1/(1+\eta)$ where $\eta = \max(-\frac{\psi(\infty)}{\psi(-\infty}, \frac{\psi(-\infty)}{\psi(\infty})$.

Definition: Let $s(x; \theta_0)$ be the score function for a nominal distribution F_0. Define

$$\tilde{\psi}(x) = [s(x; \theta_0) - a]_{-b}^{b}$$

where

$$[h(x)]_y^z = \begin{cases} y & h(x) < y \\ h(x) & y \le h(x) \le z \\ z & h(x) > z \end{cases}$$

and a is chosen to make $\tilde{\psi}$ integrate to zero wrt dF_0.

Theorem 6.3: Among all M-estimators with $\int \psi dF_0 = 0$, $\int \psi(x)s(x; \theta_0)dF_0(x) \neq 0$, and

$$\sup_x \left| \frac{\psi(x)}{\int \psi(x)s(x; \theta_0)dF_0(x)} \right| \le c,$$

that based on $\tilde{\psi}$ minimizes $\int \psi^2 dF_0 / (\int \psi s_0 dF_0)^2$. It is, up to a multiplicative constant, the unique such estimator. Among all location M-estimators with this bound on their GES this estimator maximizes the efficiency.