Saddlepoint approximation for mle in exponential families

Let X_i be iid with density

$$f_X(x) = \exp(T(x)\eta - A(\eta))d(x)$$

where $A(\eta) = \log \int \exp(T(x)\eta)d(x)\,dx$. Then

$$f_T(t) \equiv f_{T(X)}(t) = \exp(t\eta - A(\eta))c(t).$$

The mgf of T is $M_T(\phi) = \mathbf{E}(\exp(\phi T)) = \exp(A(\phi+\eta) - A(\eta))$, so the cgf is $K_T(\phi) = \log M_T(\phi) = A(\phi+\eta) - A(\eta)$. The log likelihood of a sample of size n from X (**or** T) is

$$l_n(\eta) = \eta \sum t_i - nA(\eta)$$

where $t_i = T(x_i)$. Hence the likelihood equation is $\sum t_i = nA'(\hat\eta)$, or equivalently $A'(\hat\eta) = \bar t$. For clarity we write the solution $\hat\eta(\bar t)$.

When computing a saddlepoint approximation (spa) to the distribution of $\bar T$ we get

$$f_{\bar T}(s) \doteq \left(\frac{n}{2\pi K_T{}''(\phi^*)}\right)^{\frac12} \exp(n(K_T(\phi^*) - \phi * s)).$$

Here $\phi *$ is the saddlepoint, satisfying $K_T{}'(\phi^*) = s$. Again, for clarity we write this $\phi * (s)$. Now note that $K_T{}'(\phi) = A'(\phi+\eta)$, so, since $A'(\phi * (s) + \eta) = s$ we see that $\phi * (s) + \eta$ is the mle $\hat\eta(s)$ of η, i.e., $\phi * (s) + \eta = \hat\eta(s)$, or $\phi * (s) = \hat\eta(s) - \eta$. Thus the spa is

$$f_{\bar T}(s) \doteq \left(\frac{n}{2\pi A''(\phi * (s) + \eta)}\right)^{\frac12} \exp(n(A(\phi * (s) + \eta) - A(\eta) - \phi * (s)s))$$

$$= \left(\frac{n}{2\pi A''(\hat\eta(s)}\right)^{\frac12} \exp(n(A(\hat\eta(s)) - A(\eta) - s(\hat\eta(s) - \eta)))$$

By change of variables, if $T = \sum T_i$ we get the spa

$$f_T(t) \doteq \left(2\pi A''(\hat\eta(s))\right)^{-\frac12} \exp((\eta - \hat\eta)t - nA(\eta) - A(\hat\eta))$$

$$= (2\pi J_n(\hat\eta))^{-\frac12}\,\frac{L_n(\eta)}{L_n(\hat\eta)}$$

Here $\hat\eta = \hat\eta(t/n)$, and $J_n(\eta) = -\partial^2 l_n(\eta)/\partial\eta^2 = nA''(\eta)$ in the case of a one-dimensional parameter.

Now, the transformation from T to $\hat\eta$ given by $A'(\hat\eta) = T/n$ is one-to-one with Jacobian $J_n(\hat\eta)$, so the density g of $\hat\eta$ is approximately

$$g(\hat\eta;\eta) = c\,J_n(\hat\eta)^{\frac12} L_n(\eta)/L_n(\hat\eta)$$

This formula is due to Daniels, and is called Barndorff-Nielsen's formula. Renormalization is typically needed to compute c.