Basic principle of counting

If step 1 can be performed in n_1 ways, step 2 in n_2 ways, ..., step k in n_k ways, the the ordered sequence

(step 1, step 2 ,..., step k)

can be performed in $n_1 n_2 \ldots n_k$ ways.

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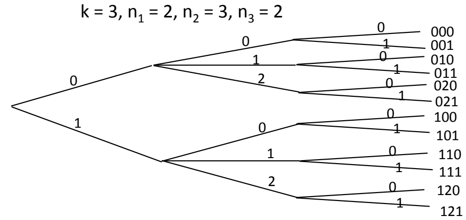
Tree diagrams

$k = 3, n_1 = 2, n_2 = 3, n_3 = 2$



$n_1 \times n_2 \times n_3 = 12$

$1 \times 3 \times 2 + 1 \times 2 \times 2 = 10$

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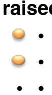
Braille alphabet

In 1824, Louis Braille invented the standard alphabet for the blind. It uses a six-dot matrix

• •
• •
• •

where some dots are raised. For example, the letter b is



The configuration with no raised dots is useless. Hence there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 - 1 = 63$ different characters.

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Permutations

An ordered sequence of k out of n distinct elements is a *permutation of length k.*

k = 2, n = 3: ab ac ba bc ca cb

Let $n! = 1 \times 2 \times \cdots \times n$ (n factorial)

The number of permutations of length k out of n distinct elements is $\frac{n!}{(n-k)!}$

The number of permutations of n distinct elements is n!   0! =

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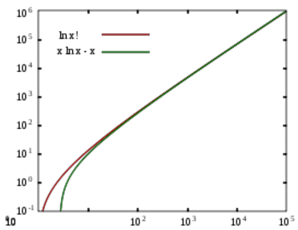
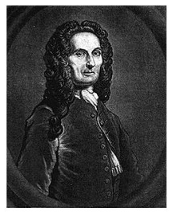
Stirling's formula

$n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}$



Originally discovered by de Moivre



n	5	25	50
n!	120	$1.551\ 10^{25}$	$3.041\ 10^{64}$
$\sqrt{2\pi}\, n^{n+1/2} e^{-n}$	180.2	$1.546\ 10^{25}$	$3.036\ 10^{64}$

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Binomial theorem

$(x+y)^2 =$

$(x+y)^3 =$

$$(x+y)^n = \sum_{i=1}^{n} c_{n,k} x^k y^{n-k}$$

$c_{n,k}$ is called a ***binomial coefficient***

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Binomial coefficient

An unordered selection of k out of n distinct objects is called a *combination* of k out of n

The number of combinations of k out of n is n choose k or

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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Binomial coefficients

$$c_{n,k} = \binom{n}{k} = \binom{n}{n-k}$$

Define 0! = 1. Then $\binom{n}{0} = \frac{n!}{0!n!} = 1$

Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Row n contains $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

Generally we have $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

85

Random numbers

The Monte Carlo method draws digits with equal probability.
P(5 consecutive digits different) =

The first 800 digits of e = 2.71828... form 160 groups of 5 each. If we arrange them into 16 batches of 10 groups each, the number of groups with five different digits are

3 1 3 4 4 1 4 4 4 2 3 1 5 4 6 3

for a total of 52 groups of five different digits, or 52/160 = 32.5%.
Are the digits of e (pseudo-) random?

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Multinomial coefficients

Let $k_1, k_2, \ldots, k_m$ be integers with $k_1+k_2+\ldots+k_m = n$.
The number of ways in which a population of n elements can be divided into m subpopulations, the first of which has k_1 elements, the second k_2 elements, etc., is

$$\frac{n!}{k_1!k_2!\cdots k_n!}$$

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Bridge hands

There are $\frac{52!}{13!13!13!13!} = 5.36 \times 10^{28}$

different bridge hands.
In how many does each player get an ace?

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Capture-recapture

To estimate the number N of goldfish in the pond at Decatur Elementary, r = 25 fishes were caught, tagged, and released. Later a second sample of n = 20 fishes were caught. Among these k = 5 were tagged. How many fishes were there in the pond?

P (catch k tagged fishes) =

How can we figure out N?

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Friday's class

Binomial coefficients
Multinomial coefficients
Capture-recapture
Problems

Problem set 3

1. (a) {(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(5,4),(5,5),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

(b) 6/21=.29

(c) All elements of the form (i,i) have probability 1/36, all other have 2/36. Total (6+15*2)/36=1

(d) Look at the relative frequency of these events and see if it is closer to the answer in (b) or in (e).

(e) 9/36=.25

2. Let people be ABC and their hats abc.

P(none get their own)=P(A gets b)P(B gets c|A gets b)+P(A gets c)P(B gets a|A gets c) = 1/3x1/2 + 1/3x1/2 = 1/3.

For the general case, it is the same as the probability that a permutation of the digits 1...n has none in their natural position.Let a_n be the number of ways this can be done.

The first individual/number has n-1 possible choices. Suppose he picks i. Then there are two options: person i picks 1, which reduces the problem to the case n-2, or not, in which case every remaining individual has precisely one forbidden choice, i.e. the case n-1. Hence $a_n = (n-1)(a_{n-2}+a_{n-1})$ with $a_1=0$, $a_2=1$. Hence $a_3=2$, $a_4=9$ etc. The probability will be $p_n=a_n/n!$, so $p_3=1/3$ (as we saw), $p_4=3/8$ etc. The general formula is

$$\sum_{i=2}^{n} \frac{(-1)^i}{i!} \to \frac{1}{e}$$

3. There are 10^5 possible choices. The highest number is ≤5 in 5^5 of those. Therefore it is ≤4 in 4^5 of them. Since {X≤5} = {X≤4} + {X=5} we get P(X=5) = $(5^5-4^5)/10^5$ = .021

4. For n=2 we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \le P(A_1) + P(A_2)$$

Hence the inequality is true when n=2. Suppose it holds for n=k-1. Then

$$P(\bigcup_{i=1}^{k} A_i) = P(\bigcup_{i=1}^{k-1} A_i \cup A_k) \le P(\bigcup_{i=1}^{k-1} A_i) + P(A_k) \le \sum_{i=1}^{k-1} P(A_i) + P(A_k) \le \sum_{i=1}^{k} P(A_i)$$

5. $P(Ace_2) = P(Ace_2|Ace_1)P(Ace_1) + P(Ace_2|Not\ Ace_1)P(Not\ Ace_1) = 3/51 \times 4/52 + 4/51 \times 48/52 = 4/52 = P(Ace_1)$

In fact, drawing the top card has the same probability of getting an ace as drawing the bottom card or any card in the (well shuffled) deck

6. $$P(K\,|\,C) = \frac{P(C\,|\,K)P(K)}{P(C\,|\,K^c)P(K^c) + P(C\,|\,K)P(K)}$$
$$= \frac{1 \times p}{\frac{1}{m}(1-p)+p} = \frac{1}{\frac{1}{m}(\frac{1}{p}-1)+1}$$

which is increasing in m and p.

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A 20-year flood

In 20 years there has been 3 disastrous flooding event at a flood plain. An embankment project can be finished in 3 years. What is the chance of another disaster before then?

Box with 20 tickets numbered 1 through 20. Draw three tickets, and let Y = highest number (worst flooding) on the tickets.

$P(Y = i) =$

$P(Y \geq 18) =$

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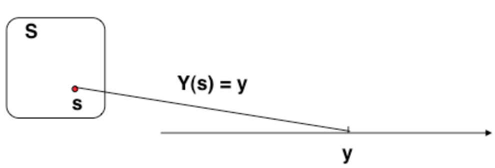
Random variables

A random variable is a function from the sample space to the reals.



Events involving a random variable have probabilities that are computed using

$$P(\{s: Y(s) = y\}) = \sum_{\{s:Y(s)=y\}} P(s)$$

in the discrete case.

Instead of writing $P(\{s: Y(s) = y\})$ we write $P(Y=y)$.

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Tay-Sachs disease

Tay-Sachs disease is a rare but fatal genetic disorder affecting mainly children of Jewish or Eastern European origin. If a couple are both carriers of the disease, their children each have probability 0.25 of being born with it. If such a couple has four children, we are interested in

X = # children with the disease

What is the sample space?

X(affa) =

$P(X = 1) =$

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Weldon's dice data

The British statistician and biologist W. R. Weldon performed, together with his wife, in the late 19 th century an experiment consisting of 26,306 throws of 12 dice.

The outcome of one die was deemed a *success* if a five or a six occurred. Thus, each trial could have between 0 and 12 successes. If the die is fair, the probability of a success is 1/3.
To get k successes we need a string of 12 S or F with exactly k S. Probability is

$$P(X=k)=\binom{12}{k}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{12-k}$$

Binomial distribution Bin (n=12,p=1/3).
As it happened, observed frequency of S was P(S) = 0.3377. Why?

The binomial distribution

X ~ Bin(n,p)
n independent trials with success probability p
X counts the number of successes

$$P(X=k)=\binom{n}{K}p^{k}(1-p)^{n-k}$$

Link

The Decatur fish pond

Recall that the probability in a sample of size n to catch X = k out of the r tagged fishes in a pond with N fishes is

Ways of choosing k tagged fishes

Ways of choosing N-k untagged

$$P(X=k)=\frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}}$$

Number of possible samples

The hypergeometric distribution

Drawing a sample of size n *without replacement* from a population of size N, where r elements are of type A and w = N-r are of type B, the number X of type A elements has distribution

$$P(X=k)=\frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}},\quad \begin{array}{c} k\le r, n-k\le N-r \\ 0\le k\le n \end{array}$$

Relations

What if we draw *with replacement*?

If N is large compared to n, does it matter whether we draw with or without replacement?

Monday's lecture

Random variable
Binomial distribution
Hypergeometric distribution

Probability mass and density functions

If a random variable X only takes on a discrete set of values (such as the integers) we call it a *discrete* random variable. It can be described by its probability mass function (pmf)

$$p_X(x) = P(X = x)$$

If the range of X contains an interval, we say that X is a *continuous* random variable. In this case we describe the random variable using its probability density function (pdf), a (piecewise) continuous curve $f_X(x)$ such that

$$P(a < X \leq b) = \int_a^b f_X(x)dx$$

WARNING!

While in the discrete case

$$p_X(x) = P(X = x),$$

this is NOT the case in the continuous case.
In fact, since

$$P(X = x) = \int_x^x f_X(u)du$$

we must have P(X = x) = 0.

The toy collector

A breakfast cereal manufacturer puts a toy in each package. There are N different toys, and they are put in packages at random in equal numbers. Let T be the number of packages needed to get at least on of each of the toys.

A_i = {no type i toy in first n packages}

$P(A_i)$ =

$P(A_i \cap A_j) =$

$P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}) =$

P(T > n) =

Link

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Voter influence

The power of a single voter in a close presidential election is *decisive* if the state has n=2k+1 voters and the rest are evenly split.
P(voter decisive) =

Average power of a voter in a state with n voters and nc electoral votes is
nc P(voter decisive)

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Waiting in line

The length of time, X, that a customer waits in line for a bank teller is described by
$f_X(x) = 0.2 \exp(- x / 5)$, $x > 0$.
However, if the customer is not waited on within 10 minutes of arrival, he will leave. What is the probability of this event?

If the customer intends to visit the bank five times in the next month, what is the probability that he will leave the line at least once?

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The cumulative distribution function

$F_X(x) = P(X \leq x)$ (cdf)

$P(X > x) =$

$P(a < X \leq b) =$

$P(X < x) =$

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Friday's lecture

Discrete and continuous random variables
Probability mass function (pmf)
Probability density function (pdf)
Cumulative distribution function (cdf)
Problems

Solutions to Problem set 4

1. There are n neighbor pairs out of $\binom{n}{2}$ pairs, so 2/(n-1) (n≥3).

2. First note that
$\{X > n+k\} \cap \{X > n\} = \{X > n+k\}$
so that
$$P(X > n+k | X > k) = \frac{P(X > n+k)}{P(X > k)}$$
Now if q = 1-p
$$P(X > k) = p\sum_{j=k+1}^{\infty} q^{j-1} = pq^k \sum_{i=0}^{\infty} q^i = q^k$$

so
$$P(X > n+k | X > n) = \frac{q^{n+k}}{q^n} = q^k = P(X > k)$$

3. If the serum has no effect the chance that any given animal gets infected is still 0.25, and if n animals get the serum,
X = # {animals getting the disease} ~ Bin(n,0.25).
P(X = 0; n = 10) = 0.056
P(X ≤ 1; n = 17) = 0.050
The n=17 result would be stronger evidence against the serum not working.

4. If the diagonal were allowed there would be 8! = 40,320 ways of putting the rooks so they cannot capture each other. Since the diagonal is not allowed, we can think of this as the problem of hats, where no one is allowed to get their own hat. The answer is then $$8!\sum_{i=0}^{8} \frac{(-1)^i}{i!} = 14{,}833$$

5. $$P(X_2 = 0) = \sum_{i=0}^{2} P(X_2 = 0 | X_1 = i) P(X_1 = i)$$
$$= 1 \times \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{2} + \tfrac{1}{4}^2 \times \tfrac{1}{4} = \tfrac{25}{64}$$

$P(X_2 > 0) = 39/64.$

$P(X_2 = 1, X_1 = 1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

and

$P(X_2 = 1) = \frac{1}{2} \times \frac{1}{2} + (2 \times (\frac{1}{2} \times \frac{1}{4}) \times \frac{1}{4} = \frac{5}{16}$

so

$P(X_1 = 1 | X_2 = 1) = \dfrac{1/4}{5/16} = \frac{4}{5}$

6. Let X = #{wins}. In a 3 game series they need to win 2, with probability $.55^2 + 2 \times .55^2 \times .45 = .57$
In a 7 game series they need to win 4, with probability
$.55^4(1 + 4 \times .45 + 10 \times .45^2 + 20 \times .45^3) = .61$
(the fifth game is not played if one team won the first four, etc.). They should want the longer series.

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Properties of the cdf

Let F(x) be a cdf. Then
(a) F is nondecreasing
(b) $F(\infty) = \lim_{x\to\infty} F(x) = 1$
(c) $F(-\infty) = 0$

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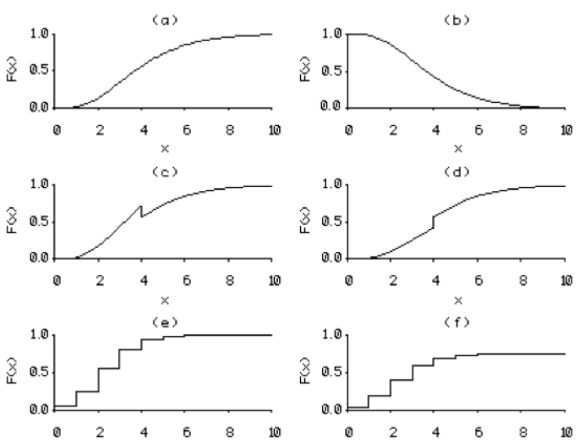
Some examples of cumulative distribution functions…or…?



Going from the cdf to the pmf/pdf

Let X be a discrete random variable. Then

$p_X(x) = F_X(x) - F_X(x-)$

If X is continuous, then

$$f_X(x) = \frac{d}{dx} F_X(x)$$

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Pareto's income distribution

If X is the annual family income, $P(X > x)$ is the proportion earning more than x.

Vilfrido Pareto, Italian economist (1848-1923) found that this can be described as $c\,x^a$, $x > x_0 > 0$, $a > 0$.

What is c?

What is the pdf?

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Change of variables

Suppose we know the distribution of X, but are interested in the distribution of $Y = f(X)$ where f is an increasing continuous function. Then

$$P(Y \le y) = P(f(X) \le y) = P(X \le f^{-1}(y)) = F_X(f^{-1}(y))$$

Does it matter whether X is discrete or continuous?

What if f is decreasing?

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The linear case

Let $Y = aX + b$ where $a > 0$. Then $F_Y(y) = P(aX + b \le y) = P(X \le (y-b)/a)$.

Discrete case

Continuous case

$a<0$

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