Monday's lecture

Properties of the cdf
Going from cdf to pmf or pdf
Change of variables

Note: Office hours next week
Bailey will do office hours on
Wednesday 2-4
Thursday 2-4
(You need to go to section on Wednesday to find out how to get into the building on Thursday)
I will do office hours on
Wednesday 9-10:30

In some of your books problem 3.4.16 has a misprint. Should be $2F_Y(a) - 1$.

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The linear case

Let Y = aX + b where a > 0. Then
$F_Y(y) = P(aX + b \le y) = P(X \le (y-b)/a)$.

Discrete case

Continuous case

a<0

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Temperature

A chemical reaction achieves a temperature, X, varying according to the pdf

$$f_X(x) = xe^{-x^2/2}, x > 0$$

where X is measured in degrees Celsius. Let Y be the temperature in degrees Fahrenheit. What is the cdf of Y?

$$F_X(x) = \int_0^x ue^{-u^2/2}\,du = 1 - e^{-x^2/2}$$

so

$$P(Y \le y) = P(1.8X + 32 \le y) = P\left(X \le \frac{y-32}{1.8}\right)$$

$$= F_X\left(\frac{x-32}{1.8}\right) = 1 - e^{-\frac{1}{2}\left(\frac{x-32}{1.8}\right)^2}$$

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A numbers game

A popular numbers game is DJ, where the winning ticket is determined from Dow Jones averages. Three sets of stocks are used: Industrials, Transportation, and Utilities, and two quotes, at 11 am and noon, Eastern time.

	11 am	Noon
I	11185.54	11188.65
T	4776.59	4785.81
U	405.08	405.49
	498 +	519 = 1 017

In this example, the winning number is 017. The payoff is 700 to 1. Suppose I bet $5. How much do I win or lose, on average?
Let p = probability my number wins =
Let X = my earnings. In the long run,
I will win $ 3500 the fraction 1/1000 of the time,
and lose $ 5 the fraction 999/1000 of the time.
The balance is

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Expected value

The expected value is a measure of the *location* of a density. For a discrete random variable it is

$$E(X) = \sum_{k=-\infty}^{\infty} k p_X(k)$$

For a continuous random variable

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

It is not always well defined.

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The Poisson distribution

This is a distribution for rare events.

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots$$

Siméon Denis Poisson (1781-1840)

E(X) =

Note that the expected value need not be a possible value!

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An exam problem

A true-false question on a statistics exam has ten parts, each worth 2 points. Any incorrect answer is penalized by -3 points, although a negative total is recorded as 0. If a student guesses, what is the expected score for the part?

Clearly, guessing should be avoided unless one is fairly certain to know the answer. What is the smallest probability that does not yield a negative expected score to a part?

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Lotka's family size model

The number of children of white families in the 1920's was described by

$$p(k) = \begin{cases} \beta p(1-p)^k, & k = 1, 2, \ldots \\ 1-\beta(1-p), & k = 0 \end{cases}$$

a *geometric distribution with modified zero term*. Here p and β are probabilities. Find the expected number of children.

For β=0.879, p=0.264, we get expected family size 2.45.

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Law of the unconscious probabilist

$$E(g(X)) = \sum_x g(x) f_X(x)$$

Proof: Suppose that g(x) takes on values $y_1, y_2, \ldots$ Then

$$P_{g(X)}(y_i) = P(g(X) = y_i) = \sum_{\{x:g(x)=y_i\}} p_X(x)$$

and

$$E(g(X)) = \sum_{i=1}^{\infty} y_i p_{g(X)}(y_i) = \sum_{i=1}^{\infty} y_i \sum_{\{x:g(x)=y_i\}} p_X(x)$$

$$= \sum_{i=1}^{\infty} \sum_{\{x:g(x)=y_i\}} g(x) p_X(x) = \sum_x g(x) p_X(x)$$

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Friday's lecture

Linear transformations
Expected values
Law of the unconscious probabilist
Problems

Nov 17: ASA Puget Sound Chapter has a dinner meeting at Bombay Grill (4737 Roosevelt NE). Tom Conlon speaks on Data Driven Life. Mixer at 6, dinner at 6:30. Student cost $5. Great opportunity to network and meet potential employers

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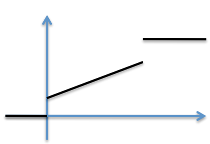
Problem solutions

1. P(X=0)=P(X=1)=1/4.
   For x>0 P(X≤x)=1/4 + x/2



2. (a) $1-F(x)=e^{-\alpha x}$ so $\lambda(x)=\alpha$
   (b) $\frac{f(x)}{1-F(x)} = -\frac{d}{dx}\log(1-F(x))$

so $-\log(1-F(x)) = \int_0^t \lambda(x)dx$

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(c) $\lambda_s(t) = 2\lambda_n(t)$

P(40 yo nonsmoker raches 60)

$= P(\text{nonsmoker life} > 60 \mid \text{nonsmoker life} > 40)$

$$= \frac{1-F_n(60)}{1-F_n(40)} = \frac{\exp(-\int_0^{60}\lambda_n(x)dx)}{\exp(\int_0^{40}\lambda_n(x)dx)}$$

$= \exp(-\int_{40}^{60}\lambda_n(x)dx$

P(40 yo smoker reaches 60)

$= \exp(-\int_{40}^{60}\lambda_s(x)dx) = \exp(-2\int_{40}^{60}\lambda_n(x)dx$

$= \left[\exp(\int_{40}^{60}\lambda_n(x)dx)\right]^2$

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3. (a)

$$P(D \le t) = \frac{\pi t^2}{\pi 1^2} = t^2 \text{ so } f_D(t) = 2t, 0 \le t \le 1$$

(b) $E(D) = \int_0^1 t(2t)\,dt = \frac{2}{3}$

4. (a) $P(-X=k)=P(X=-k)=p_X(-k)$

(b)

$$p(X^+ = k) = \begin{cases} p_X(k), k > 0 \\ \sum_{j=-\infty}^{0} p_X(j), k \le 0 \end{cases}$$

(c) $P(X^- = k) = P((-X)^+ = k)$

(d)

$$P(\operatorname{sgn} X = k) = \begin{cases} \sum_{k>0} p_X(k), k = 1 \\ p_X(0), k = 0 \\ \sum_{k<0} p_X(k), k = -1 \end{cases}$$

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5 $P(-\frac{\log X}{\alpha} \le y) = P(X \ge e^{-\alpha y}) = 1 - e^{-\alpha y}$

since $F_X(x)=x$, $0 \le x \le 1$

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Review

Experiment
Outcomes
Sample space
Events
Venn diagrams
Probability axioms
Probability facts
Conditional probability
Theorem of total probability
Probability trees
Bayes theorem

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Randomized response survey
Independence
Counting rules
Binomial coefficients
Multinomial coefficients
Capture-recapture
Random variable
Binomial distribution
Hypergeometric distribution
Discrete and continuous random variables
Probability mass function (pmf)
Probability density function (pdf)
Cumulative distribution function (cdf)
Properties of the cdf
Going from cdf to pmf or pdf
Change of variables
Linear transformations
Expected values
Law of the unconscious probabilist

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A betting game

You bet \$1 that a fair coin will show heads on the first toss. If it does, you win \$1. If not, you bet \$2 that it will show heads on the second toss. If it does, you win \$2-\$1=\$1. If not, you bet \$4 that it will show heads on the third toss. If it does, you win \$4-\$2-\$1=\$1, otherwise you lose \$1+\$2+\$4=\$7. Is this a fair game?

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Linear transformation

$$E(aX+b)=\int(ax+b)f_X(x)\,dx$$
$$=a\int xf_X(x)\,dx+b\int f_X(x)\,dx$$
$$=aE(X)+b$$

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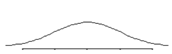
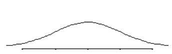
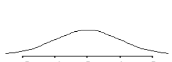
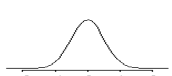
Measuring spread

The expected value is one measure of the *location* of the distribution of a random variable. In addition to the location it is useful to have a description of how spread out a distribution is.

different location
same spread

same location
different spread

$Var(X)=E(X-E(X))^2$ — **moment of inertia in mechanics**
units: those of X squared

$sd(X)=\sqrt{Var(X)}$ — **units of X**

Computation

$$Var(X)=E(X^2)-2E(XE(X))+[E(X)]^2$$
$$=E(X^2)-2E(X)E(X)+[E(X)]^2$$
$$=E(X^2)-[E(X)]^2$$

The variance is defined when $E(X^2)<\infty$

Var(aX + b) =

Some examples

Exponential distribution $f(x) = \alpha e^{-\alpha x}, x > 0$

$$E(X) = \int_0^\infty x\alpha \exp(-\alpha x)dx = \frac{1}{\alpha}$$

$$E(X^2) = \int_0^\infty x^2\alpha \exp(-\alpha x)dx = \frac{2}{\alpha^2}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

Poisson distribution: $p(x) = \lambda^x e^{-\lambda} / x!$

$E(X) = \lambda$

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1)\frac{\lambda^x}{x!}e^{-\lambda} = \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}e^{-\lambda} = \lambda^2$$

$$E(X^2) = E(X(X-1)) + E(X) = \lambda^2 + \lambda$$

$$Var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

The Bienaymé-Chebyshev inequality

Consider a random variable X with expected value μ and variance σ^2. Then for any $t > 0$ we have that

$$P(|X - \mu| > t) \le \frac{\sigma^2}{t^2}$$

Proof:

$$P(|X - \mu| > t) = \int_{\{x:|x-\mu|>t\}} f(x)dx$$

$$\le \int_{\{x:|x-\mu|>t\}} \frac{(x-\mu)^2}{t^2} f(x)dx \le \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{t^2} f(x)dx = \frac{\sigma^2}{t^2}$$

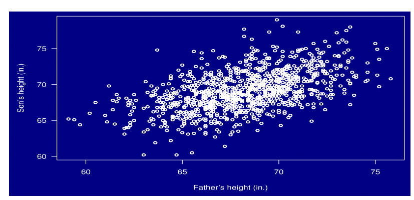
How good is the Bienaymé-Chebyshev inequality?

Let X ~ Exp(1). How well does the inequality estimate
P(|X - 1|>2)?
We have E(X)=Var(X)=1, so the inequality says that
P(|X - 1|/1 > 2/1) ≤

while the exact probability is

P(|X - 1| > 2) =

Heights of fathers and sons



Heights of fathers and sons, from Galton. There is a tendency for tall fathers to have tall sons. Therefore it is not sufficient to describe only the distribution of fathers' heights and sons' heights separately: we need the *joint* behavior of the two random variables.

Joint distributions

For two discrete random variables we define the *joint probability mass function* as

$p_{X,Y}(x,y)=P(X=x ,Y=y)$

The *joint cdf* is (for discrete, continuous, or mixed random variables)

$F_{X,Y}(x,y) = P(X \le x , Y \le y)$

For two continuous random variables, the *joint pdf* is given by

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

$$P((X,Y) \in C) =$$

Queue lengths

A supermarket has two express lines. At a given time, let X and Y be the number of customers in the first and second line, respectively. The joint distribution is

X

Y	0	1	2	3
0	.1	.2	0	0
1	.2	.25	.05	0
2	0	.05	.05	.025
3	0	0	.025	.05

How do we find $P(X \neq Y)$?

Monday's lecture

Variance
Chebyshev's inequality
Bivariate distributions

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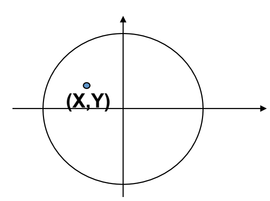
A random point in a circle



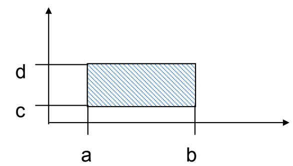
What is the density of (X,Y) if we choose the point at random in the unit circle, in the sense that equal areas have equal probabilities?

The distribution of a ratio

Let $f_{X,Y}(x,y) = \exp(-(x+y))$, $x>0$, $y>0$. Find the cdf of $Z=X/Y$.

Computing probabilities from the joint cdf

Given $F_{X,Y}(x,y)$, how do we compute $P(a<X\leq b, c<Y\leq d)$?

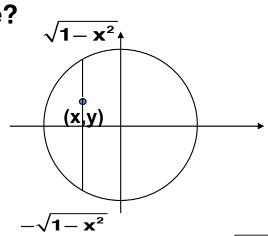


We see that the sought probability is given by

$$F_{X,Y}(b,d) - F_{X,Y}(b,c) - F_{X,Y}(a,d) + F_{X,Y}(a,c)$$

More about a random point in a circle

Given a point chosen at random in the unit circle, what is the distribution of the x-coordinate?



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy$$

$$= \frac{2}{\pi}\sqrt{1-x^2}, -1 \leq x \leq 1$$

Marginal distributions

If (X,Y) is discrete, the marginal distribution of X is given by

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

while if they are continuous it is given by

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)dy$$

The marginal distribution contains no information about the joint behavior of X and Y.