Linear difference equations with constant coefficients

1. The forward shift operator

Many probability computations can be put in terms of recurrence relations that have to be satisfied by successive probabilities. The theory of *difference equations* is the appropriate tool for solving such problems. This theory looks a lot like the theory for linear differential equations with constant coefficients.

In order to simplify notation we introduce the *forward shift operator* E, that takes a term u_n and shifts the index one step forward to u_{n+1}. We write

$$Eu_n = u_{n+1}$$

$$E^2 u_n = EEu_n = Eu_{n+1} = Eu_{n+2}$$

$$\cdots$$

$$E^r u_n = u_{n+r}$$

The general linear difference equation of order r with constant coefficients is

$$\Phi(E)u_n = f(n) \tag{1}$$

where $\Phi(E)$ is a polynomial of degree r in E and where we may assume that the coefficient of E^r is 1.

2. Homogeneous difference equations

The simplest class of difference equations of the form (1) has $f(n) = 0$, that is simply

$$\Phi(E)u_n = 0.$$

These are called *homogeneous* equations.

When $\Phi(E) = (E - \lambda_1)(E - \lambda_2)\cdots(E - \lambda_r)$ where the λ_i are constants that are all distinct from each other, one can prove that the most general solution to the homogeneous equation is

$$u_n = a_1\lambda_1^n + a_2\lambda_2^n + .. + a_r\lambda_r^n$$

where $a_1, a_2, \cdots, a_r$ are arbitrary constants.

When $\Phi(E)$ contains a repeated factor $(E - \lambda_\alpha)^h$, the corresponding part of the general solution becomes

$$\lambda_\alpha^n(a_\alpha + a_{\alpha+1}n + a_{\alpha+2}n^{(2)} + \cdots + a_{\alpha+h-1}n^{(h-1)})$$

where $n^{(k)} = n(n-1)(n-2)\cdots(n-k+1) = n!/(n-k)!$.

In order to find the n'th term of a linear difference equation of order r, one can of course start by r initial values, and the solve recursively for any given n. Thus, if we want our solution to satisfy certain initial conditions we may first determine the general solution, and then (if possible) make it satisfy the initial conditions. There can be no more than r such initial conditions, but they need not (as when we compute the solution recursively) necessarily be conditions on $u_0, \ldots, u_{r-1}$, but can be on any set of r values.

Example 1. Solve $u_{n+2} - u_n = 0$.

The equation can be written in the form

$$(E^2 - 1)u_n = 0$$

or

$$(E - 1)(E + 1)u_n = 0$$

The general solution is therefore

$$u_n = a(-1)^n + b1^n$$

where a, b, c are constants.

Example 2. Find the general solution to the equation

$$u_{n+4} - 9u_{n+3} + 30u_{n+2} - 44u_{n+1} + 24u_n = 0$$

and hence obtain the particular solution satisfying the conditions

$$u_0 = 1,\ u_1 = 5,\ u_2 = 1,\ u_3 = -45.$$

The equation may be written in the form

$$(E^4 - 9E^3 + 30E^2 - 44E + 24)u_n = 0$$

$$(E-2)^3(E-3)u_n = 0.$$

The general solution is therefore

$$u_n = 2^n(a + bn + cn(n-1)) + d3^n$$

where a, b, c, d are constants.

For the particular side conditions we have

$$u_0 = a + d = 1,$$

$$u_1 = 2a + 2b + 3d = 5$$

$$u_2 = 4a + 8b + 8c + 9d = 1$$

$$u_3 = 8a + 24b + 48c + 27d = -45$$

whence $a = 0$, $b = 1$, $c = -2$, $d = 1$, so the particular solution is

$$u_n = 2^n n(3 - 2n) + 3^n.$$

3. Non-homogeneous difference equations

When solving linear differential equations with constant coefficients one first finds the general solution for the homogeneous equation, and then adds any particular solution to the non-homogeneous one. The same recipe works in the case of difference equations, i.e. first find the general solution to

$$\Phi(E)u_n = 0$$

and a particular solution to

$$\Phi(E) = f(n)$$

and add the two together for the general solution to the latter equation. Thus to solve these more general equations, the only new problem is to identify some particular solutions. We will only give a few examples here, not attempting to treat this problem in any generality.

(i) $f(n) = k\mu^n,\ \ \mu \neq \lambda_i, i = 1, 2, \ldots, r$

In this case one can show that

$$u_n = \frac{k\mu^n}{\Phi(\mu)}$$

is a particular solution to $\Phi(E)u_n = k\mu^n$. Let namely $\Phi(E) = \sum a_i E^i$. Then

$$\Phi(E)\,\frac{k\mu^n}{\Phi(\mu)} = \frac{\sum a_i E^i k\mu^n}{\sum a_i \mu^i} = k\,\frac{\sum a_i \mu^{n+i}}{\sum a_i \mu^i} = k\mu^n$$

Example 3. The general solution of

$$u_{n+2} - 5u_{n+1} + 6u_n = 3(4^n)$$

is $u_n = a2^n + b3^n + \frac{3}{2}4^n$ where a and b are arbitrary constants.

(ii) $f(n) = k\mu^n$, $\mu = \lambda_i$**,** λ_i **a non-repeated factor of** $\Phi(E)$

In this case a particular solution is given by

$$\frac{kn\mu^{n-1}}{\Phi'(\mu)}$$

where $\Phi'(\mu)$ denotes $\left(\frac{d}{dE}\Phi(E)\right)_{E=\mu}$.

Example 4. The general solution of

$$u_{n+2} - 5u_{n+1} + 6u_n = 3(2^n)$$

is

$$u_n = a2^n + b3^n + \frac{3n2^{n-1}}{-1} = (a - \frac{3n}{2})2^n + b3^n$$

where a, b are arbitrary constants.

(iii) $f(n) = k\mu^n$, $\mu = \lambda_i$**,** λ_i **a repeated factor of** $\Phi(E)$

Suppose now that $(E - \lambda_i)$ is repeated h times in $\Phi(E)$. Then

$$\frac{kn^{(h)}\mu^{n-h}}{\Phi^{(h)}(\mu)},$$

where $n^{(k)} = n(n-1)\cdots(n-k+1)$, is a particular solution of the equation $\Phi(E)u_n = k\mu^n$.

Example 5. The general solution of the equation

$$(E-2)^3(E-3)u_n = 5(2^n)$$

is

$$u_n = (a + bn + cn(n-1))2^n + d3^n + \frac{5n(n-1)(n-2)2^{(n-3)}}{-6}$$

with a, b, c, d are arbitrary constants

(iv) $f(n)$ **is a polynomial in** n

First write f as a polynomial in the factorial powers $n^{(k)}$, so

$$f(n) = a_0 + a_1 n + a_2 n^{(2)} + \cdots$$

Now define the *difference operator* Δ by $\Delta u_n = u_{n+1} - u_n = (E-1)u_n$. Using the symbolic relationship $E = 1 + \Delta$ we can re-express $\Phi(E)$ as $\Psi(\Delta)$. Still arguing symbolically, a particular solution is obtained by

$$u_n = \frac{1}{\Phi(E)} f(n) = \frac{1}{\Psi(\Delta)} f(n),$$

provided that we can make any sense out of $\frac{1}{\Psi(\Delta)}$. The way this will be done is by expanding $\frac{1}{\Psi(\Delta)}$ in powers of Δ and using long division. The following rules are needed:

$$\Delta n^{(r)} = rn^{(r-1)}$$

$$\Delta^2 n^{(r)} = r^{(2)} n^{(r-2)}$$

$$\cdots$$

$$\Delta^k n^{(r)} = r^{(k)} n^{(r-k))} \text{ for } k \le r,$$

$$= 0 \text{ for } k > r$$

and

$$\Delta^{-1} n^{(r)} = \frac{n^{(r+1)}}{r+1}$$

$$\Delta^{-2} n^{(r)} = \frac{n^{(r+2)}}{(r+1)(r+2)}$$

$$\cdots$$

$$\Delta^{-k} n^{(r)} = \frac{n^{(r+k)}}{(r+1)(r+2)\cdots(r+k)}.$$

Example 6. Find a particular solution of the equation

$$u_{n+2} - 7u_{n+1} + 12u_n = 3n^2 + 2n + 2.$$

First write $3n^2 + 2n + 2 = 3n^{(2)} + 5n^{(1)} + 2$ and $E^2 - 7E + 12 = (2 - \Delta)(3 - \Delta)$. Thus we get

$$u_n = \frac{1}{(2-\Delta)(3-\Delta)}(3n^{(2)} + 5n^{(1)} + 2)$$

$$= \frac{1}{6}\left(1 - \frac{\Delta}{2}\right)^{-1}\left(1 - \frac{\Delta}{3}\right)^{-1}(3n^{(2)} + 5n^{(1)} + 2)$$

$$= \frac{1}{6}(1 + \frac{\Delta}{2} + \frac{\Delta^2}{4} + \cdots)(1 + \frac{\Delta}{3} + \frac{\Delta^2}{9} + \cdots)(3n^{(2)} + 5n^{(1)} + 2)$$

$$= \frac{1}{6}(1 + \frac{5}{6}\Delta + \frac{19}{36}\Delta^2 + \cdots)(3n^{(2)} + 5n^{(1)} + 2)$$

$$= \frac{1}{2}n^{(2)} + \frac{5}{6}n^{(1)} + \frac{1}{3} + \frac{5}{6}n^{(1)} + \frac{25}{36} + \frac{19}{36}$$

$$= \frac{1}{2}n^2 + \frac{7}{6}n + \frac{14}{9}$$

Example 7. Find a particular solution of the equation

$$u_{n+3} - 5u_{n+2} + 7u_{n+1} - 3u_n = n^2 + 4n + 1.$$

The required solution is

$$u_n = \frac{1}{\Delta^2(\Delta - 2)}(n^{(2)} + 5n^{(1)} + 1)$$

$$= -\frac{1}{2}\Delta^{-2}(1 + \frac{\Delta}{2} + \frac{\Delta^2}{4} + \frac{\Delta^3}{8} + \frac{\Delta^4}{16} + \cdots)(n^{(2)} + 5n^{(1)} + 1)$$

$$= -\frac{1}{2}(\Delta^{-2} + \frac{1}{2}\Delta^{-1} + \frac{1}{4} + \frac{1}{8}\Delta + \frac{1}{16}\Delta^2 + \cdots)(n^{(2)} + 5n^{(1)} + 1)$$

$$= -\frac{1}{2}\left(\frac{n^{(4)}}{12} + \frac{n^{(3)}}{6} + \frac{1}{4}n^{(2)} + \frac{2}{8}n^{(1)} + \frac{2}{16} + \frac{5n^{(3)}}{6} + \frac{5n^{(2)}}{4} + \frac{5}{4}n^{(1)} + \frac{5}{8} + \frac{n^{(2)}}{2} + \frac{1}{2}n^{(1)} + \frac{1}{4}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{12}n^4 + \frac{1}{2}n^3 - \frac{1}{12}n^2 + \frac{3}{2}n + 1\right).$$