

Stat/Math 491  
Homework 4 (due 11/7)

1. Suppose we have iid observations from a mixture of exponential distributions with

$$\text{density } f(x; \theta) = \sum_{j=1}^k p_j \lambda_j e^{-x \lambda_j}.$$

(a) Let  $\Lambda$  be a random variable taking on the value  $\lambda_j$  with probability  $p_j, j = 1, \dots, k$ . Show that the density above can be written as an expected value with respect to  $\Lambda$ .

(b) Describe the Gibbs sampler for the expected value in (a).

2. The following is a simple model for an epidemic process. Consider an urn with  $N$  balls,  $R$  of which are red and  $B$  black. A ball is drawn at random and replaced by  $1+d$  balls of the same color.

(a) Interpret the cases  $d = 0$  and  $d = -1$ .

(b) Show that the probability of drawing a red ball at the  $(n+1)$ th draw is  $(R+rd)/(N+nd)$  where  $r$  is the number of red draws in the first  $n$  draws.

(c) Show that the total number of red balls drawn in the first  $n$  draws is a Markov chain, and determine its transition probabilities.

(d) Determine the marginal distribution of the chain in (c).

3. Let  $Q$  be a transition matrix, not necessarily symmetric, and assume that the target function for an MCMC algorithm satisfies  $f(x) > 0$  for all  $x$ .

Define  $\alpha(x, y) = \min \left\{ \frac{f(y)Q(y, x)}{f(x)Q(x, y)}, 1 \right\}$  when  $f(x)Q(x, y) > 0$ , and 1 otherwise. Let  $X_n = x$ ,

and generate a candidate point  $Y = y$  from the distribution  $Q(x, \bullet)$ . With probability  $\alpha(x, y)$  we set  $X_{n+1} = y$  and with the complementary probability we set  $X_{n+1} = x$ .

(a) Show that the resulting process  $X_n$  is a Markov chain with stationary distribution  $f$ .

(b) Show that the Metropolis algorithm results when  $Q$  is symmetric.