Solution to group exercise, Oct 10.

(a) The first step can either be up or down. The probability of reaching 0 before N if we are at k+1 is the same as that probability if we had started from k+1 by the Markov property. Hence $u_k = P(X_T = 0 \mid X_0 = k)$

$$= pP(X_T = 0 \mid X_1 = k, X_0 = k) + qP(X_T = 0 \mid X_1 = k-1, X_0 = k)$$

$$= pu_{k+1} + qu_{k-1}$$

$u_0 = 1$ and $u_N = 0$ (since if we start at N we are immediately absorbed at N and cannot reach 0).

This is called a *first step analysis*.

(b) Write $u_2 = \frac{1}{p}(u_1 - qu_0)$ so $x_2 = u_2 - u_1 = \left(\frac{1}{p} - 1\right)u_1 - \frac{q}{p}u_0 = \frac{q}{p}(u_1 - u_0) = \frac{q}{p}x_1$.

Similarly $x_k = u_k - u_{k-1} = \frac{q}{p}(u_{k-1} - u_{k-2}) = \frac{q}{p}x_{k-1} = \cdots = \left(\frac{q}{p}\right)^{k-1} x_1$

(c) $\sum_{k=1}^{N} x_k = \sum_{k=1}^{N}(u_k - u_{k-1}) = u_N - u_1 = -1$. But from (b) we also have that

$$\sum_{k=1}^{N} x_k = \sum_{k=1}^{N}\left(\frac{p}{q}\right)^{k-1} x_1 = x_1\left(1 + \frac{p}{q} + \cdots \left(\frac{p}{q}\right)^{N-1}\right) \text{ so } x_1 = -\frac{1}{1 + \frac{p}{q} + \cdots \left(\frac{p}{q}\right)^{N-1}}.$$

Since $u_k - 1 = \sum_{i=1}^{k} x_i$ the formula follows.

(d) When $p/q = 1$ we get $u_k = k/N$.

(e) $P(X_T = N \mid X_0 = k) = 1 - P(X_T = 0 \mid X_0 = k) = 1 - u_k$

(f) Using a first step analysis again, with $m_k = E(T \mid X_o = k)$ we get $m_k = 1 + pm_{k+1} + qm_{k-1}$. As above, let $x_k = m_k - m_{k-1}$. Then

$m_{k+1} - m_k = m_k\left(\frac{q}{p}\right) - \frac{1}{p} - \frac{q}{p}m_{k-1}$ so $x_{k+1} = \frac{q}{p}x_k - \frac{1}{p}$. For k = 2 we get $x_2 = \frac{q}{p}x_1 - \frac{1}{p}$.

For k = 3 we have $x_3 = \frac{q}{p}\left(\frac{q}{p}x_1 - \frac{1}{p}\right) - \frac{1}{p} = \left(\frac{q}{p}\right)^2 x_1 - \frac{q}{p^2} - \frac{1}{p}$ and

generally $x_k = \left(\frac{q}{p}\right)^{k-1} x_1 - \sum_{j=1}^{k-1}\frac{q^{j-1}}{p^j}$. Since $\sum_{k=1}^{N} x_k = m_N - m_0 = 0$ we

get $x_1 \sum_{jk=1}^{N}\left(\frac{q}{p}\right)^{k-1} = \sum_{k=1}^{N}\sum_{j=1}^{k-1}\frac{q^{j-1}}{p^j}$ or $x_1 = \frac{\sum_{k=1}^{N}\sum_{j=1}^{k-1}\frac{q^{j-1}}{p^j}}{\sum_{k=1}^{N}\left(\frac{q}{p}\right)^{k-1}}$, whence $x_n = \frac{\sum_{k=n}^{N}\sum_{j=1}^{k-1}\frac{q^{j-1}}{p^j}}{\sum_{k=1}^{N}\left(\frac{q}{p}\right)^{k-1}}$ and $m_1 = x_1$,

$m_2 = x_2 + m_1 = x_2 + x_1$ etc. For the special case $p = q$ we get that $m_k = k(N-k)$.