Completeness and Basu's theorem

Definition: Let T be a statistic. It is **complete** if for all bounded functions g we have $E_\theta g(T) = 0 \Rightarrow g = 0$ a.s.

Note: Completeness is a property of the distribution of T, rather than of the statistic itself.

Example: Let **X** follow a natural exponential family with int(Θ) non-empty. Recall that the density of the natural sufficient statistic $T(\mathbf{X})$ is of the form

$$f_T(t;\theta) = a(\theta)b(t)\exp(\theta \cdot t)$$

Assume that $\int g(t)a(\theta)b(t)\exp(\theta \cdot t)dt = 0$. Define $d\nu^\pm(t) = g^\pm(t)b(t)dt$ where $g^\pm(t) = \max\{\pm g(t), 0\}$. Then $\int \exp(\theta \cdot t)\, d\nu^+(t) = \int \exp(\theta \cdot t) d\nu^+(t)$, so by the uniqueness theorem for Laplace transforms $\nu^+ = \nu^-$, whence $g(t) = -g(t)$ so $g(t) = 0$.

Definition: A statistic U is **conditionally ancillary** given a statistic V if $\mathcal{L}(U|V)$ is free of θ.

Example: If $U = X$ then V is sufficient, while if V is constant then U is ancillary.

Theorem 1.2: If V is complete, there are no non-trivial conditionally ancillary statistics given V.

Proof: Let A be a set defined in terms of U. Compute

$$\int (P(A \mid V = v) - 1_A(v)) f_V(v;\theta)dv = E_\theta(P(A \mid V = v)) - P_\theta(A) = P_\theta(A) - P_\theta(A) = 0$$

By completeness $P(A \mid V) = 1_A$ a.s., so U must be a constant a.s.

Theorem 1.3: Let T, U, V be statistics. Suppose that (T,V) is sufficient, and that U is conditionally ancillary, given V. If the distribution of (T,V) is complete, then T and U are conditionally independent, given V.

Corollary (Basu's theorem): Suppose T is sufficient, and U is ancillary. Then if T is complete, T and U are independent.

Proof: $\int \{P(U \le u \mid T = t, V = v) - P(U \le u \mid V = v)\} f_{T,V}(t,v;\theta)dtdv$

$$= E_\theta P(U \le u \mid T,V) - E_\theta P(U \le u \mid V) = P_\theta(U \le u) - P_\theta(U \le u) = 0$$

so by the completeness the conditional distribution of U given (T,V) is the same as the conditional distribution of U given V,