Spatial point processes

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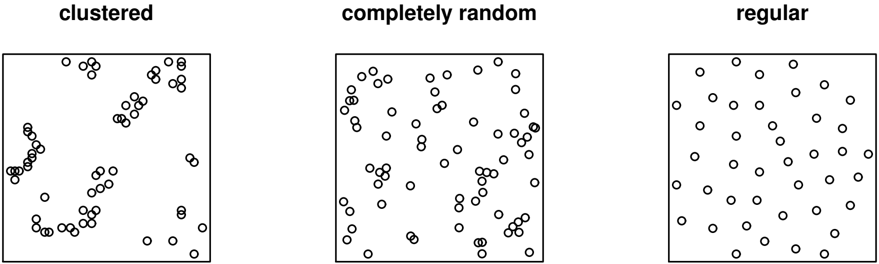
Definition

- A point process N is a stochastic mechanism or rule to produce point patterns or realisations according to the distribution of the process.
- A marked point process is a point process where each point x_i of the process is assigned a quantity $m(x_i)$, called a mark. Often, marks are integers or real numbers but more general marks can also be considered.
- A spatio-temporal point process is a random collection of points, where each point represents the time and location of an event. Even marks can be included.

Spatial point patterns

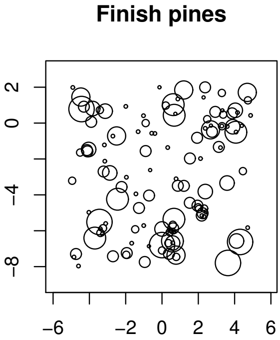


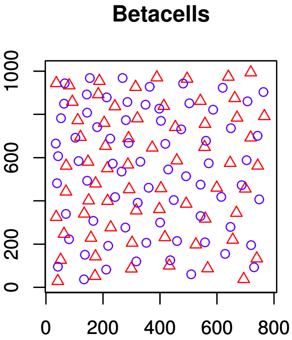
Examples

- Locations of betacells within a rectangular region in a cat's eye (regular)
- Locations of Finnish pine saplings (clustered)
- Locations of Spanish towns (regular)
- Locations of galaxes (clustered)

Remark: Very different scales, from microscopic to cosmic

Marked point patterns





Finnish pine saplings: locations and diameters

Beta-type retina cells in the retina of a cat: locations and type (red triangles "on", blue circles "off")

Spatio-temporal point patterns

- Points (x_i, t_i) represent events of zero (or negligible) duration at random instants t_i at random locations x_i: occurences of earthquakes
- Points appear randomly, remain in the pattern for a random length of time and subsequently disappear: forest fires, plant communities
- Points are moving in space with or without interaction: physical particles, animals or storm centers

Remarks

Remark 1: A point process N can be regarded either as a counting measure ($N(B)$ is the random number of points in B) or as a random set (random collection of points).

Remark 2: We assume that all point processes are locally finate ($N(B) < \infty$ for all bounded B) and simple (there are no multiple points ($x_i \neq x_j$ if $i \neq j$)).

Remark 3: To avoid confusion between points of the process and points of $\mathbb{R}^d$ (here, $d = 2$), the points of the process or a point pattern (realization) are called events.

First-order properties (without marks)

The mean number of points of N in B is $\mathbb{E}(N(B))$ (depends on the set B). Furthermore,

$$\Lambda(B) = \mathbb{E}(N(B))$$

and Λ is called the intensity measure.

Under some continuity conditions, a density function λ, called the intensity function, exists, and

$$\Lambda(B) = \int_B \lambda(x)\, dx.$$

Some properties of point processes: stationarity and isotropy

- A point process N is stationary (translation invariant) if N and the translated point process N_x have the same distribution for all translations x, i.e.

$$N = \{x_1, x_2, ...\} \text{ and } N_x = \{x_1 + x, x_2 + x, ...\}$$

  have the same distribution for all $x \in \mathbb{R}^d$.
- In the stationary case $\lambda(x) = \lambda$. $0 < \lambda < \infty$ is the mean number of points of N per unit area and is called intensity.
- A point process is isotropic (rotation invariant) if its characteristics are invariant under rotations, i.e.

$$N = \{x_1, x_2, ...\} \text{ and } rN_x = \{rx_1, rx_2, ...\}$$

  have the same distribution for any rotation r around the origin.

Some summary functions

1. Let D_1 denote the distance from an arbitrary event to the nearest other event. Then, the nearest neighbour distance function is

$$G(r) = P(D_1 \leq r)$$

2. Let D_2 denote the distance from an arbitrary point to the nearest event. Then,

$$F(r) = P(D_2 \leq r)$$

3. Using G and F we can define the so-called J function as

$$J(r) = \frac{1 - G(r)}{1 - F(r)}$$

(whenever $F(r) > 0$)

Second-order properties (stationary and isotropic processes)

- Ripley's K function (Ripley, 1977)

  $\lambda K(r) = \mathbb{E}[\#$ further events within distance r of an arbitrary event$]$.

- A variance stabilizing and centered version of the K function (Besag, 1977) is used, namely

$$L(r) = \sqrt{K(r)/\pi}$$

  or

$$L(r) - r = \sqrt{K(r)/\pi} - r$$

  (the latter equals 0 under CSR).

- Ripley's K function is a cumulative version of the pair-correllation function g, namely $g(r) = K'(r)/2\pi r$

Note on estimation of summary functions

- Typically, a point pattern is observed in a (bounded) observation window and points outside the window are not observed.
- Estimators (except for $J(r)$) need to be edge-corrected
- Edge correction methods include plus sampling, minus sampling, Ripley's isotropic correction and translation (stationary) correction
- Pair-correlation function is typically estimated by kernel estimation

More complicated situations

The summary statistics (except F) have been generelized into more complicated situations

- Inhomogeneous case (under certain type of non-stationarity)
- Anisotropic case (the summary function can be estimated separately in different directions)
- Marked case
- Spatio-temporal case

Poisson process

A point process is a homogeneous Poisson process (CSR) if

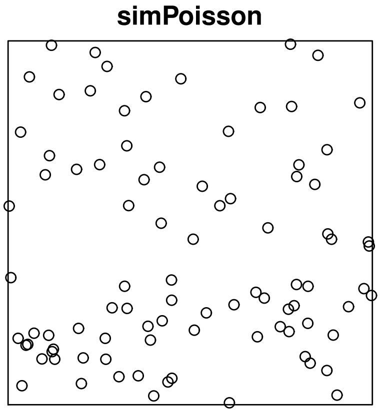
(P1) for some $\lambda > 0$ and any finite region B, $N(B)$ has a Poisson distribution with mean $\lambda|B|$

(P2) given $N(B) = n$, the events in B form an independent random sample from the uniform distribution on B

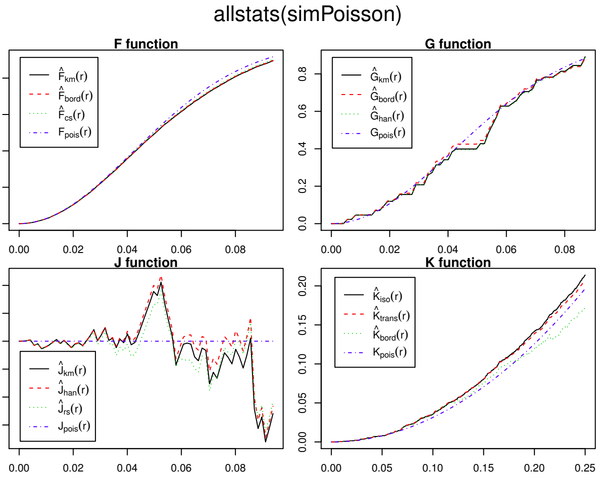
For a homogeneous Poisson process, the analytic form of the summary functions mentioned earlier are known:

- $G(r) = F(r) = 1 - \exp(-\lambda\pi r^2)$
- $J(r) = 1$
- $K(r) = \pi r^2$, $L(r) = r$ and $L(r) - r = 0$
- $g(r) = 1$

Realization of a homogeneous Poisson process with intensity 100



Summary statistics



Non-constant intensity

- Inhomogeneous Poisson process: intensity λ (in homogeneous Poisson process) replaced by an intensity function $\lambda(x)$
- Cox process (doubly stochastic process): intensity surface is random
- log Gaussian Cox process: Cox process where the logarithm of the intensity surface is a Gaussian process

Cluster processes: Matérn cluster process

Cluster processes are models for aggregated spatial point patterns

For Matérn cluster process

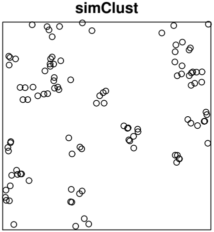
(MC1) parent events form a Poisson process with intensity λ

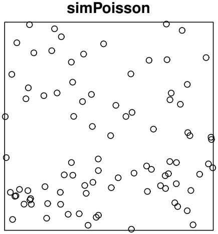
(MC2) each parent produces a random number S of daughters (offsprings), realized independently and identically for each parent according to some probability distribution

(MC3) the locations of the daughters in a cluster are independently and uniformly scattered in the disc of radius R centered at the parent point.

The cluster process consists only of the daughter points.

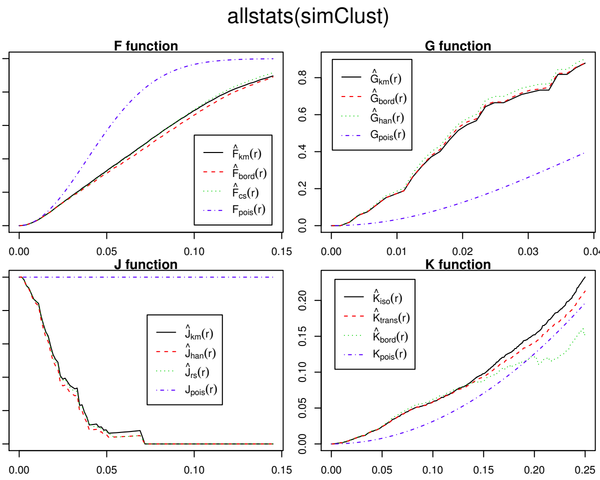
Realization of a Matérn cluster process





Left: Parent point intensity 20, cluster radius 0.05, average number of daughter points per cluster 5
Right: Poisson process with intensity 100

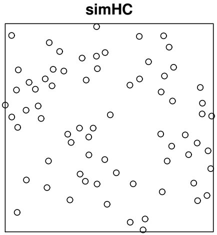
Summary statistics

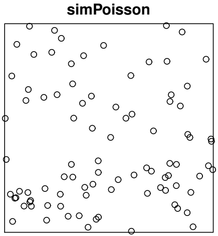


Hard-core processes: Matérn I hard-core process

- Hard-core processes are models for regular spatial point patterns
- There is a minimum allowed distance, called hard-core distance, between any two points
- Matérn I hard-core process: A Poisson process with intensity λ is thinned by delating all pairs of points that are at distance less than the hard-core radius apart.

Realization of a Matérn I hard-core process

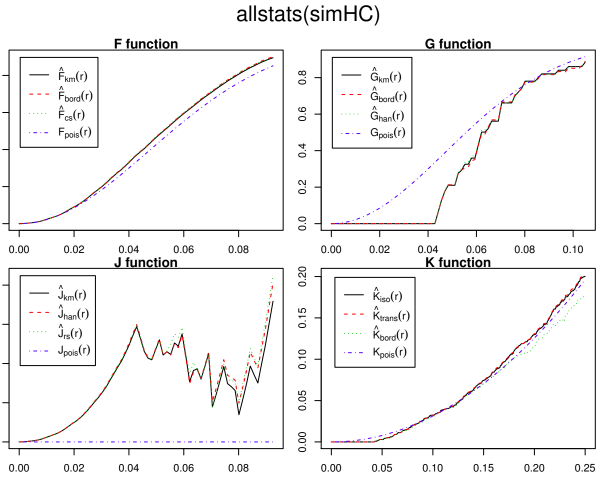




Left: Hard-core process with the initial Poisson intensity 300, hard-core radius 0.04

Right: Poisson process with intensity 100

Summary statistics



Markov point process (finate)

- Models for point patterns with interaction between the events. Can model both attraction (clustering) and inhibition (regularity).
- There is interaction between events if they are "neighbours", for example, if they are close enough to each other.
- The neighbourhood of x_i is the set of points that are neighbours to x_i
- Each process (defined on an arbitrary but fixed finate region B) is characterized by a density function w.r.t. a Poisson process.

Definitions

Definition 1: The Papangelou conditional intensity is defined as

$$\lambda(x_i|\mathbf{x} \setminus \{x_i\}) = \frac{f(x_1, ..., x_n)}{f(\mathbf{x} \setminus \{x_i\})}$$

for $x_i \in \mathbf{x}$. It can be regarded as the conditional probability of having an event in an infinitesimal region around x_i given that the rest of the process is $\mathbf{x} \setminus \{x_i\}$

Definition 2: A point process is a Markov point process (with a given neighbourhood) if the Papangelou conditional intensity of $x_i \in \mathbf{x}$ depends only on the configuration in the neighbourhood of x_i.

Pairwise interaction processes: Strauss process

- Models for inhibition/regularity. Interaction only between pairs of events included.
- Two points are neighbours if they are closer than distance R apart
- The density function is given by

$$f(\mathbf{x}) = \alpha\beta^{n(\mathbf{x})}\gamma^{s(\mathbf{x})}, \ \beta > 0, \ \gamma > 0,$$

  where
  - $\beta > 0$ is the effect of a single event (connected to the intensity of the process)
  - $0 < \gamma \leq 1$ is an interaction parameter
  - $n(\mathbf{x})$ is the number of points in the configuration
  - $s(\mathbf{x})$ is the number of R close pairs in the configuration, where $R > 0$ is an interaction radius (range of interaction)
  - α is a normalizing constant

Saturation process

The density function is given by

$$f(\mathbf{x}) = \alpha \beta^{n(\mathbf{x})} \prod_{x_i \in \mathbf{x}} \gamma^{\min\{s_{x_i}(\mathbf{x}), m\}}, \ \beta > 0, \ \gamma > 0,$$

where α, β, γ, and $n(\mathbf{x})$ are as in the case of Strauss process and

- $s_{x_i}(\mathbf{x})$ is the number of R close neighbours of x_i in the configuration x
- m is the saturation threshold
- If the strength of interaction is
  - $\gamma = 1$, we have a Poisson process
  - $0 < \gamma < 1$, we have a regular process, and
  - $\gamma > 1$, we have a clustered process

Spatio-temporal point processes

- Can be characterized by a conditional rate process $\lambda(x, t)$, where x is the spatial location and t is the time point.
- $\lambda(x, t)$ can be thought as the frequency with which events are expected to occur around a particular location (x, t) in space-time, conditionally on the prior history of the point process up to time t.

Example: self-exciting point process

- Often used in epidemiology and seismology, and even in criminology, to model events that are clustered together in time and space
- The conditional rate process can be written as

$$\lambda(x, t) = \mu(x, t) + \sum_i \nu(x - x_i, t - t_i),$$

where the sum is over all points (x_i, t_i) with $t_i < t$, and μ and ν represent the background rate and clustering density (may depend on marks), respectively.

In seismology (earthquakes)

- the magnitude of the earthquake can be added as a mark
- earthquakes divided into background events and aftershock events
- the background events can occur independently according to a stationary Poisson process
- each earthquake elevates a risk of aftershocks and the elevated risk spreads in space and time according to the kernel ν
- the kernel ν is chosen such that the elevated risk increases with earthquake magnitude and decreases in space and time away from each event

Goodness-fit and residuals

- Goodness-of-fit of any model can be checked by comparing the values of some summary statistic estimated from the data with those estimated from simulated realizations of the fitted model.
- Local versions of summary statistics (LISA) and values of conditional intensity can be used to find outliers
- Baddeley *et al.* (2005) define residuals for point process models and propose diagnostic plots based on them. These residuals apply to any point process model that has a conditional intensity.

Final remark

- The choice of the model class should be based on the application and on what we know about the phenomenon under investigation
- For example, we have introduced three glasses of models for clustering
  - Cox processes, inhomogeneous Poisson processes
    $\rightarrow$ appropriate when environmental heterogeneity
  - cluster processes
    $\rightarrow$ e.g. large trees and seedlings
  - Markov processes (e.g. saturation process)
    $\rightarrow$ appropriate when attraction between the events

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