Multivariate Localization Methods in Ensemble Kalman Filtering

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Ensemble Kalman Filter

- $\mathbf{x}_t \in \mathbb{R}^n$: unobservable state at time t
  $\mathbf{y}_t \in \mathbb{R}^p$: observations at time t
  Nonlinear system eq. $\mathbf{x}_t = \mathcal{M}(\mathbf{x}_{t-1}) + \mathbf{e}_t$, where $\mathbf{e}_t \sim N_n(\mathbf{0}, \mathbf{Q}_t)$
  Observation eq. $\mathbf{y}_t = \mathbf{H}_t \mathbf{x}_t + \epsilon_t$, where $\epsilon_t \sim N_p(\mathbf{0}, \mathbf{R}_t)$
- M-member background ensemble: $\{\mathbf{x}^{b(k)} : k = 1, \ldots, M\} \in \mathbb{R}^{n \times M}$
  background mean $\bar{\mathbf{x}}^b = \frac{1}{M} \sum_{k=1}^{M} \mathbf{x}^{b(k)}$
  background error covariance $\mathbf{P}^b = \frac{1}{M-1} \sum_{k=1}^{M} \mathbf{X}^{b(k)} [\mathbf{X}^{b(k)}]^T$, where $\mathbf{X}^{b(k)} = \mathbf{x}^{b(k)} - \bar{\mathbf{x}}^b$
- analysis mean $\bar{\mathbf{x}}^a = \bar{\mathbf{x}}^b + \mathbf{K}(\mathbf{y} - \mathbf{H}\bar{\mathbf{x}}^b)$
  analysis covariance $\mathbf{P}^a = (\mathbf{I} - \mathbf{KH})\mathbf{P}^b$
- Kalman gain $\mathbf{K} = \mathbf{P}^b \mathbf{H}^T (\mathbf{H}\mathbf{P}^b\mathbf{H} + \mathbf{R})^{-1}$

Multivariate Localization

- Localization (covariance tapering) reduces the sampling errors, often by Schur (elementwise) product of $\mathbf{P}^b$ and a localization matrix $\mathbf{C}$ given by a compactly supported correlation function (Hamill et al. 2001; Houtekamer and Mitchell 2002):

$$\tilde{\mathbf{P}}^b = \mathbf{P}^b \circ \mathbf{C}.$$

- Applying univariate localization directly to multiple state variables causes *rank deficiency* problem.
- If $\boldsymbol{\rho}(\cdot) = \{\rho_{ij}(\cdot)\}_{i,j=1,\ldots,N}$ is a correlation function giving a localization matrix, where $N =$ the number of state variables, then not only each $\rho_{ij}(\cdot)$ but also $\boldsymbol{\rho}(\cdot)$ must be a valid function for correlation.

Localization for $N = 2$

1. Use $\rho_{ij}(\cdot) = \beta_{ij}\ \rho(\cdot)$ with $|\beta_{ij}| < 1$, $|\beta_{ji}| < 1$, and $\beta_{ii} = \beta_{jj} = 1$.
   - $\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$ is positive-definite and of full rank for any β with $|\beta| < 1$.
2. Use multivariate compactly supported correlation functions.
   - Bivariate Askey function (Porcu et al. 2012)

$$\rho_{ij}(d; \nu, c) = \beta_{ij} \left(1 - \frac{d}{c}\right)_{+}^{\nu+\mu_{ij}},$$

     where $c > 0$, $\mu_{12} = \mu_{21} \geq \frac{1}{2}(\mu_{11} + \mu_{22})$, $\nu \geq [\frac{1}{2}s] + 2$, and s is space dimension.
   - $|\beta_{ij}| \leq \frac{\Gamma(1+\mu_{12})}{\Gamma(1+\nu+\mu_{12})}\sqrt{\frac{\Gamma(1+\nu+\mu_{11})\Gamma(1+\nu+\mu_{22})}{\Gamma(1+\mu_{11})\Gamma(1+\mu_{22})}}$, $\beta_{ii} = \beta_{jj} = 1$
   - $|\beta_{ij}| \leq 1$ if $\mu_{11} = \mu_{22}$.

Bivariate Lorenz Model (Lorenz 95)

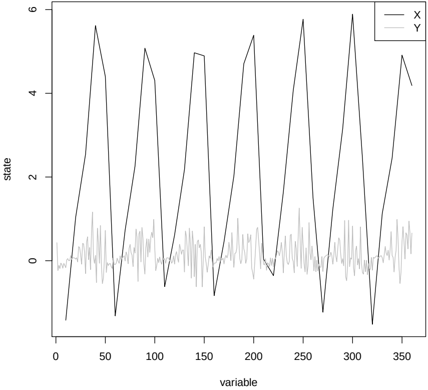
- X_k and $Y_{j,k}$ are equally spaced on a latitude circle ($j = 1, \ldots, J$ and $k = 1, \ldots, K$).
- With boundary conditions $X_{k\pm K} = X_K$, $Y_{j,k\pm K} = Y_{j,k}$, $Y_{j-J,k} = Y_{j,k-1}$, and $Y_{j+J,k} = Y_{j,k+1}$,

$$\frac{dX_k}{dt} = -X_{k-1}(X_{k-2} - X_{k+1}) - X_k - (ha/b)\sum_{j=1}^{J} Y_{j,k} + F,$$

$$\frac{dY_{j,k}}{dt} = -abY_{j+1,k}(Y_{j+2,k} - Y_{j-1,k}) - aY_{j,k} + (ha/b)X_k$$

Bivariate Lorenz Model

- 36 variables of X, 360 variables of Y, $a = 10$, $b = 10$, $h = 2$



longitudinal profiles

$\mathbf{P}^b$ for $N = 2$

- Bivariate ensembles $\mathbf{x}^{b(k)} = (\mathbf{x}_1^{b(k)}, \mathbf{x}_2^{b(k)})$, $k = 1, \ldots, M$
- $\mathbf{P}^b$ is expressed as

$$\mathbf{P}^b = \begin{pmatrix} \mathbf{P}_{11}^b & \mathbf{P}_{12}^b \\ \mathbf{P}_{21}^b & \mathbf{P}_{22}^b \end{pmatrix},$$

where $\mathbf{P}_{ij}^b = \frac{1}{M-1}\mathbf{X}_i^b\mathbf{X}_j^{bT}$ and $\mathbf{X}_i^b = \mathbf{x}_i^{b(k)} - \bar{\mathbf{x}}_i^{b(k)}$.

Experiments

- Two scenarios for observation
  1. Observe 20% of $\mathbf{X}$ and 90% of $\mathbf{Y}$ at locations where $\mathbf{X}$ is not observed.
  2. Fully observe $\mathbf{X}$ and $\mathbf{Y}$ as a control group.
- Four localization schemes
  - S1 uses $\mathbf{P}^b$.
  - S2 $\mathbf{P}^b_{12} = \mathbf{P}^b_{21} = \mathbf{0}$.
  - S3 localizes $\mathbf{P}^b_{11}$ and $\mathbf{P}^b_{22}$, but $\mathbf{P}^b_{12} = \mathbf{P}^b_{21} = \mathbf{0}$. $(\beta = 0)$
  - S4 localizes $\mathbf{P}^b_{11}$, $\mathbf{P}^b_{22}$, $\mathbf{P}^b_{12}$, $\mathbf{P}^b_{21}$. $(0 < \beta < 1)$

Localization S4

- Apply the following two correlation functions:
  1. Gaspari-Cohn function: $\rho_{ij}(d;c) = \beta_{ij}\rho(d;c),\ i,j = 1,2$, where

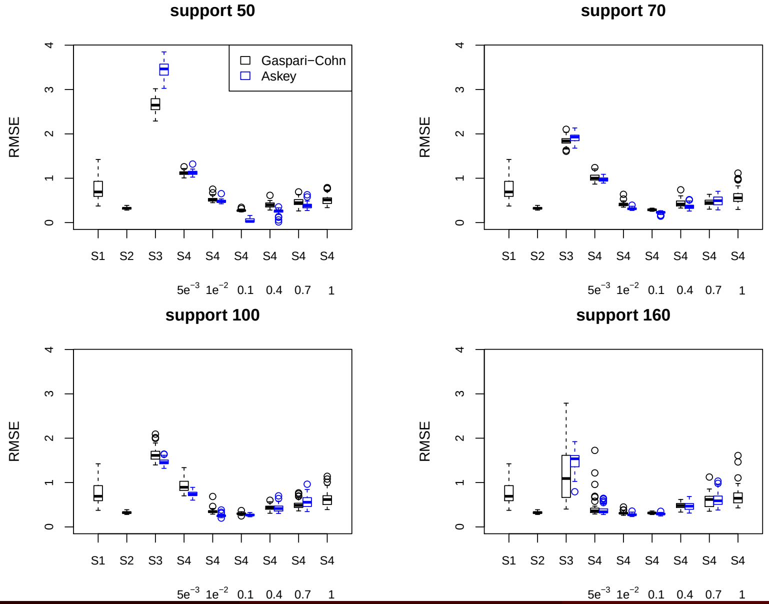
$$\rho(d;c) = \begin{cases} -\frac{1}{4}(|d|/c)^5+\frac{1}{2}(d/c)^4+\frac{5}{8}(|d|/c)^3-\frac{5}{3}(d/c)^2+1, & 0\leq|d|\leq c; \\ \frac{1}{12}(|d|/c)^5-\frac{1}{2}(d/c)^4+\frac{5}{8}(|d|/c)^3+\frac{5}{3}(d/c)^2-5(|d|/c)+4-\frac{2}{3}c/|d|, & c\leq|d|\leq 2c; \\ 0, & 2c\leq|d| \end{cases}$$

     and $\beta_{11} = \beta_{22} = 1$, $0 \leq \beta_{ij} \leq 1$. (support=2c)
  2. Bivariate Askey function

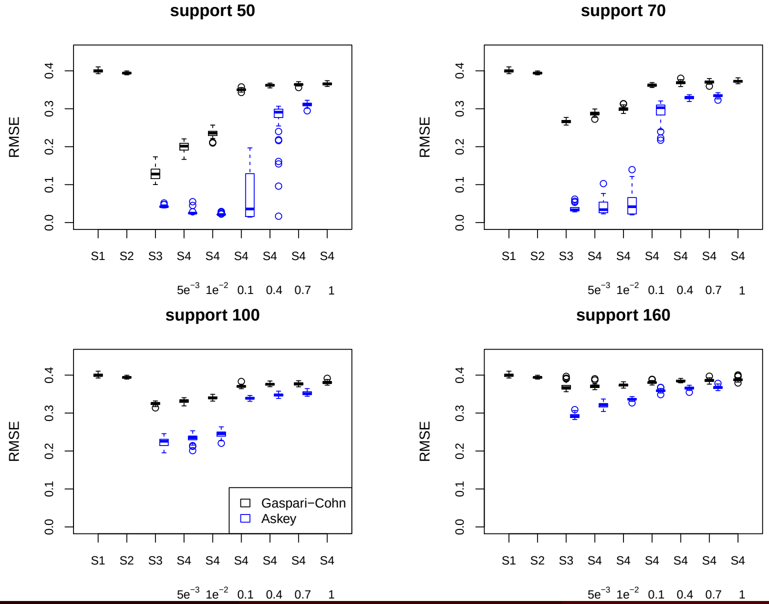
$$\rho_{ij}(d;c) = \beta_{ij}\left(1 - \frac{|d|}{c}\right)_{+}^{\nu+\mu_{ij}},\ i,j = 1,2$$

     with $\mu_{11} = 0$, $\mu_{22} = 2$, $\mu_{ij} = 1$, $\nu = 3$, and $\beta_{11} = \beta_{22} = 1$, $0 \leq \beta_{ij} \leq 0.7$. (support=$c$)

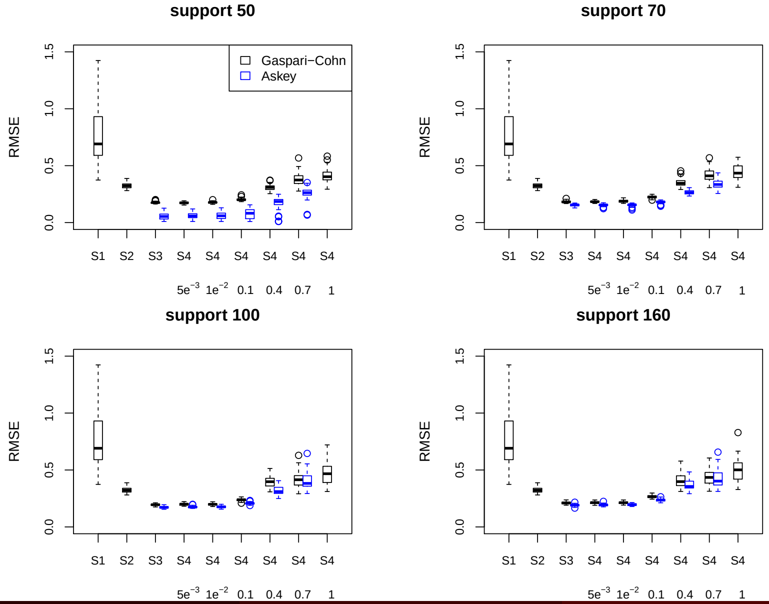
Results for **X** in scenario 1



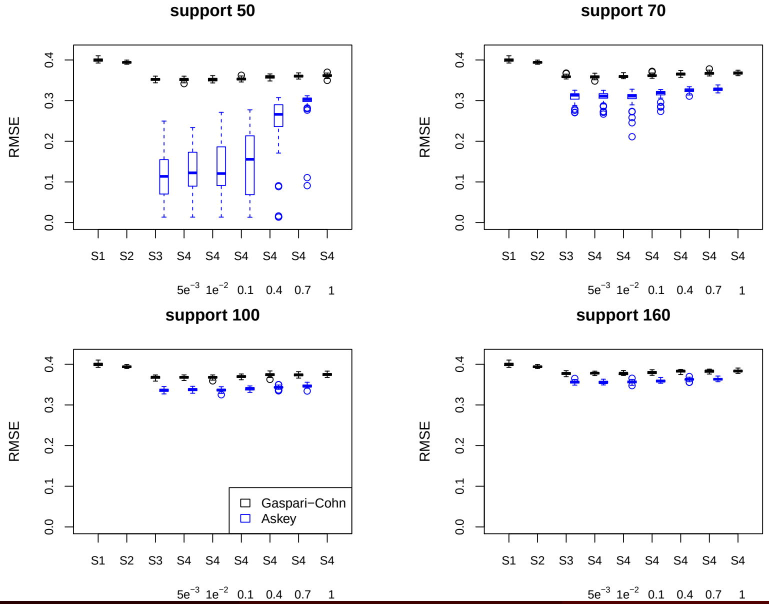
Results for **Y** in scenario 1



Results for **X** in scenario 2



Results for **Y** in scenario 2



Future Work

- Find optimal β at each time step?
- Use different localization length for each state variable?

References

- Gaspari, G. and Cohn, S. E. (1999). Construction of correlation functions in two and three dimensions. *Q. J. R. Meteorol. Soc.*, 125, 723-757.
- Hamill, T. M., Whitaker, J. S., and Snyder, C. (2001). Distance-dependent filtering of background error covariance estimates in an ensemble Kalman filter. *Mon. Wea. Rev.*, 129, 2776-2790.
- Houtekamer, P. L. and Mitchell, H. L. (1998). Data assimilation using an ensemble Kalman filter technique. *Mon. Wea. Rev.*, 126, 796-811.
- Lorenz, E. N. (1995). Predictability-A problem partly solved. Proceedings of the Seminar on Predictability, vol. 1. ECMWF, Reading, Berkshire, UK.
- Porcu, E., Daley, D. J., Buhmann, M. and Bevilacqua, M. (2012). Radial basis functions with compact support for multivariate geostatistics. *Stoch. Environ. Res. Risk. Assess.*, 27, 909-922.