# Bayesian Inference for Poisson Line Cluster Point Processes

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Joint work with:

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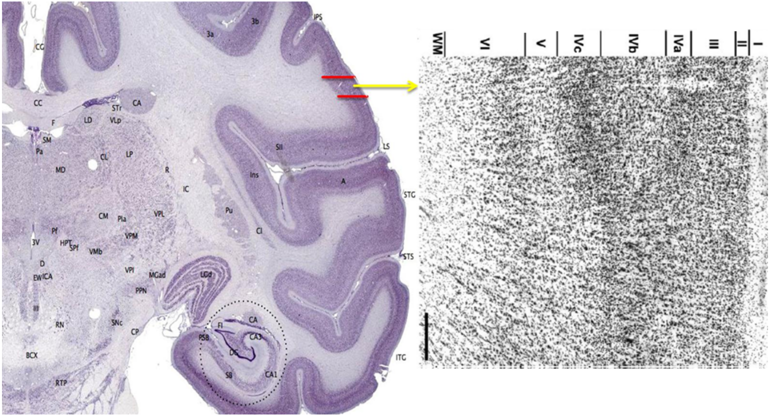
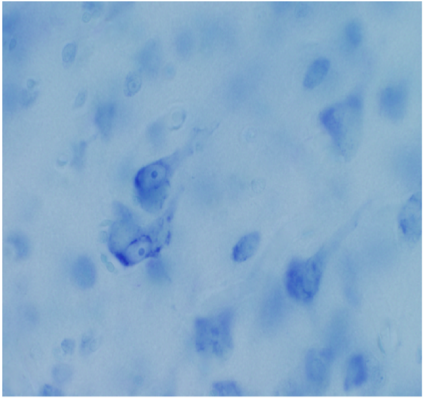
# Minicolumns; To Be or Not?
# A columnar anisotropy in 3D point patterns



# Minicolumns; From Reality to Availability!

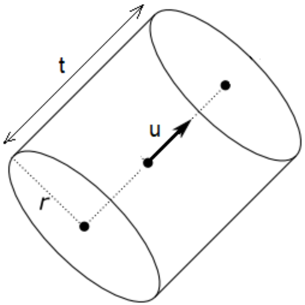


The Real World ⟶ The Artificial World

# Outline

1. Cylindrical $K$-function
   - Definition
   - Non-parametric Estimation
   - Edge effects
   - Application
2. Poisson Line Cluster Point Processes (PLCPP)
   - Definition
   - Intensity and rose of directions
   - Moments
   - Densities for the PLCPP and a fnite version of the PLCPP
   - Simulation based Bayesian inference

# Cylindrical $K$-function: Definition



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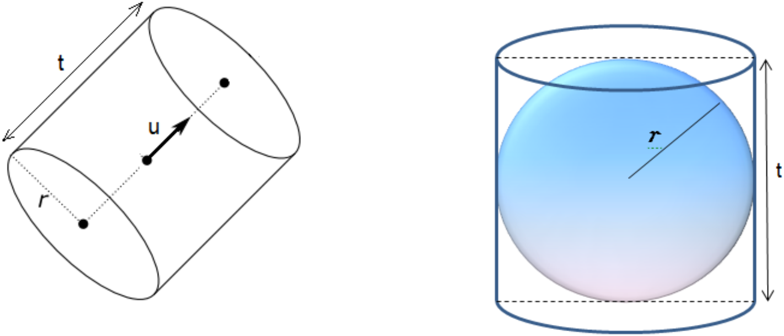


Cylindrical $K$-function vs Spherical (Ripley's) $K$-function

# Definition

- $W \subset \mathbb{R}^d$: an arbitrary Borel set with $0 < |W| < \infty$
- For $\mathbf{u} \in \mathbb{S}_+^{d-1}$, the *cylindrical $K$-function in the directions* $\pm\mathbf{u}$ is defined as

$$K_{\mathbf{u}}(r,t) = \frac{1}{\rho^2 |W|} \mathrm{E} \sum_{\mathbf{x}_1,\mathbf{x}_2 \in \mathbf{X}}^{\neq} \mathbf{1}[\mathbf{x}_1 \in W, \mathbf{x}_2 - \mathbf{x}_1 \in C_{\mathbf{u}}(r,t)], \quad r,t > 0$$

  - $C(r,t)$: the $d$-dimensional cylinder with midpoint $\mathbf{o}$, radius $r > 0$, and height $2t > 0$ in the direction along the $x_d$-axis.

# Properties

- $\rho K_u(r, t)$: the mean number of further points in the cylinder with radius $r$, height $t$, direction $u$, and midpoint at $\xi$, conditional on that $X$ has a point at the location $\xi$

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- Does not depend on the choice of $W$ because of stationarity of $X$
- For $d = 3$, a relationship with the space-time $K$-function in Diggle *et al* (2009).

# Non-parametric Estimation

$$\hat{K}_{\mathbf{u}}(r,t) = \frac{1}{\widehat{\rho^2}} \sum_{\mathbf{x}_1,\mathbf{x}_2 \in \mathbf{y}}^{\neq} w(\mathbf{x}_1,\mathbf{x}_2)\mathbf{1}[\mathbf{x}_2 - \mathbf{x}_1 \in C_{\mathbf{u}}(r,t)]$$

- $\widehat{\rho^2} = n(n-1)/|W|^2$ is the usual estimate of $\rho^2$

# Edge effects

Three possibilities:

- Translation correction factor
- Isotropic correction factor
- Combined correction factor

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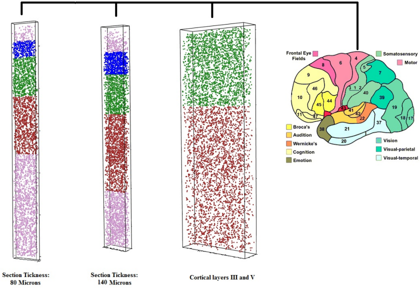
**Our choice: the translation correction factor** because:

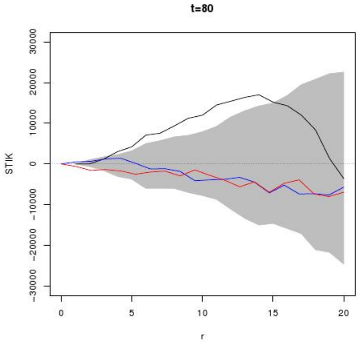
- no restriction on the shape of $W$.
- no restriction on the directions of $C_{\mathbf{u}}(r, t)$.
- similar results for the translation and the combined results based on the simulations

# Data sets




- $|W| = 510\mu \times 138\mu \times 518\mu$
- Layer III; Brodmann area 4

# Application in neuroscience: 3D Minicolumn data




- For $t = 80$, n=999, type I error probability= 0.026

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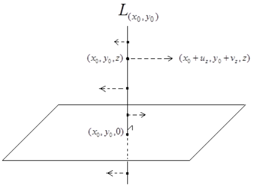


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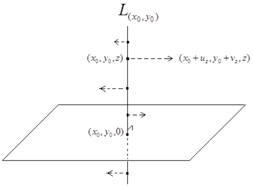


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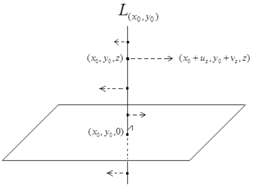


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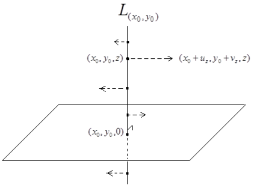


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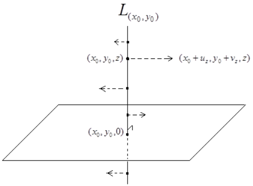


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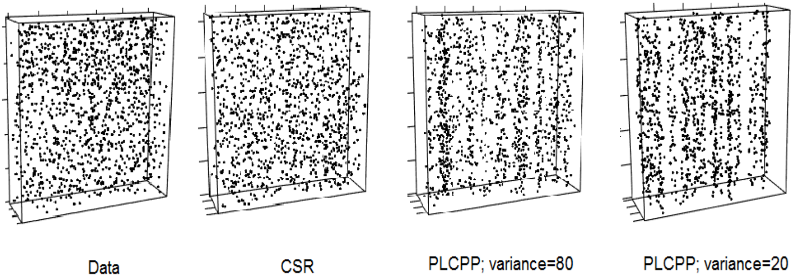
# Realizations of PLCPP



# Notation

- **The phase representation:**
  - $\mathbf{L}$: is identified by a point process $\mathbf{\Phi} \subset H \times \mathbb{S}^{d-1}$
    - $H$: the hyperplane perpendicular to the $x_d$-axis
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- $p_{\mathbf{u}^\perp}(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{u})\mathbf{u}$: the orthogonal projection of $\mathbf{x} \in \mathbb{R}^d$ onto $\mathbf{u}^\perp$ (the hyperplane perpendicular to $\mathbf{u}$ and containing the origin).

# PLCPP: Assumptions

- $\mathbf{\Phi}$: a Poisson process with intensity measure $\beta\lambda(\mathrm{d}\mathbf{y})M(\mathrm{d}\mathbf{u})$
  - $\beta$: a positive and finite parameter
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# PLCPP: Intensity and rose of directions

- **Useful for computational reason:**

  to specify the distribution of the Poisson line process $\mathbf{L}$ by $(\beta, M)$.

- **Useful for interpretation:**

  to specify the distribution of the Poisson line process $\mathbf{L}$ by $(\rho_L, \mathcal{R})$
  - $\rho_L$: the intensity of $\mathbf{L}$
  - $\mathcal{R}$: the rose of directions of $\mathbf{L}$

- $\rho_L$: the mean length of lines in $\mathbf{L}$ within any region of unit volume in $\mathbb{R}^d$
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- $\mathcal{R}$: the distribution for the direction of a typical line in **L**

**A one-to-one correspondence between $(\beta, M)$ and $(\rho_L, \mathcal{R})$:**
For any Borel set $B \subseteq \mathbb{S}^{d-1}$, [0.2]

$$\rho_L = \beta \int 1/|u_d|\, M(\mathrm{d}\mathbf{u}), \quad \mathcal{R}(B) = \int_B 1/|u_d|\, M(\mathrm{d}\mathbf{u}) \Big/ \int 1/|u_d|\, M(\mathrm{d}\mathbf{u}).$$

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# PLCPP: Moments

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$$\rho = \mathrm{E}[\Lambda(\mathbf{o})], \quad \rho^2 g(\mathbf{x}) = \mathrm{E}[\Lambda(\mathbf{o})\Lambda(\mathbf{x})], \quad \mathbf{x} \in \mathbb{R}^d.$$

We verified that

$$\rho = \alpha \rho_L$$

and

$$g(\mathbf{x}) = 1 + \frac{1}{\rho_L} \int k_{\mathbf{u}^\perp} * \tilde{k}_{\mathbf{u}^\perp}(p_{\mathbf{u}^\perp}(\mathbf{x}))\, \mathcal{R}(\mathrm{d}\mathbf{u}), \quad \mathbf{x} \in \mathbb{R}^d,$$

Thus $g > 1$, reflecting the clustering of the PLPCP

# PLCPP: Densities

- $\mathbf{X}_W = \mathbf{X} \cap W$, the PLCPP restricted to a bounded region $W \subset \mathbb{R}^d$
- A finite approximation of the latent process $\Phi$

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$$f(\mathbf{u}|\boldsymbol{\mu}, \kappa) = c_d(\kappa)\exp(\kappa\boldsymbol{\mu}\cdot\mathbf{u}), \quad c_d(\kappa) = \frac{\kappa^{d/2-1}}{(2\pi)^{d/2} I_{d/2-1}(\kappa)}, \quad \mathbf{u} \in \mathbb{S}^{d-1},$$

$I_d$: the modified Bessel function of the first kind and order $d$

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$$f(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}|\mathbf{\Phi}, \alpha, \sigma^2) = \exp\left(|W| - \int_W \Lambda(\mathbf{x}|\mathbf{\Phi}, \alpha, \sigma^2)\,\mathrm{d}x\right) \prod_{i=1}^{n} \Lambda(\mathbf{x}_i|\mathbf{\Phi}, \alpha, \sigma^2)$$

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- We replace the infinite $\mathbf{\Phi}$ by a finite approximation $\mathbf{\Phi}_S = \mathbf{\Phi} \cap S$ so that

$$\Lambda(\mathbf{x}|\mathbf{\Phi}_S, \alpha, \sigma^2) = \alpha \sum_{(\mathbf{y},\mathbf{u}) \in \mathbf{\Phi}_S} f(p_{\mathbf{u}^\perp}(\mathbf{x} - \mathbf{y})|\sigma^2)$$

  is a finite sum.

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- $S$: the set of lines hitting a bounded region $W_{\text{ext}} \supseteq W$

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- **The choice of $W_{\text{ext}}$**

  - depends on the model and the data at hand
  - to elliminate boundary effects, $W_{\text{ext}}$ is sufficiently large so that it is very unlikely that for some line $l_i = l(\mathbf{y}_i, \mathbf{u}_i) \in \mathbf{L}$ with $(\mathbf{y}_i, \mathbf{u}_i) \notin S$, $\mathbf{X}_i$ has a point in $W$.

# PLCPP: Densities

- $\mathbf{\Phi}_S$ is a Poisson process on $S$ with intensity function

$$\chi(\mathbf{y}, \mathbf{u}|\boldsymbol{\mu}, \kappa) = \rho_L |u_d| f(\mathbf{u}|\boldsymbol{\mu}, \kappa)$$

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with respect to the measure $\lambda(\mathrm{d}\mathbf{y})\nu_{d-1}(\mathrm{d}\mathbf{u})$

- The reference process corresponds to the case of an isotropic Poisson line process with unit intensity.

# PLCPP: Densities

- Letting $\mathbf{\Phi}_{0,S} = \mathbf{\Phi}_0 \cap S$, then the density of $\mathbf{\Phi}_S$ with respect to the distribution of $\mathbf{\Phi}_{0,S}$ is

$$f(\{(\mathbf{y}_1, \mathbf{u}_1), \ldots, (\mathbf{y}_k, \mathbf{u}_k)\} | \rho_L, \boldsymbol{\mu}, \kappa)$$
$$= \exp\left(\int_S \left[\chi_0(\mathbf{y}, \mathbf{u}) - \chi(\mathbf{y}, \mathbf{u} | \boldsymbol{\mu}, \kappa)\right] \lambda(\mathrm{d}\mathbf{y}) \nu_{d-1}(\mathrm{d}\mathbf{u})\right) \prod_{j=1}^{k} \frac{\chi(\mathbf{y}_j, \mathbf{u}_j | \boldsymbol{\mu}, \kappa)}{\chi_0(\mathbf{y}_j, \mathbf{u}_j)}$$

for finite point configurations $\{(\mathbf{y}_1, \mathbf{u}_1), \ldots, (\mathbf{y}_k, \mathbf{u}_k)\} \subset S$.

# PLCPP: Densities

- That is
$f(\{(\mathbf{y}_1, \mathbf{u}_1), \dots, (\mathbf{y}_k, \mathbf{u}_k)\} | \rho_L, \boldsymbol{\mu}, \kappa)$

$$\propto \exp\left(-\rho_L \int_{\mathbb{S}^{d-1}} |u_d| \lambda(J_{\mathbf{u}}) f(\mathbf{u}|\boldsymbol{\mu}, \kappa) \nu_{d-1}(\mathrm{d}\mathbf{u})\right)$$
$$\times \prod_{j=1}^{k} \left[\frac{2\pi^{d/2}}{\Gamma(d/2)} \rho_L f(\mathbf{u}_j|\boldsymbol{\mu}, \kappa) \mathbf{1}[\mathbf{y}_j \in J_{\mathbf{u}_j}]\right]$$

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- The missing data: $\mathbf{\Phi}_S$
- The joint density of $\mathbf{X}_W$ and $\mathbf{\Phi}_S$:

$$
\begin{aligned}
& l(\alpha, \sigma^2, \rho_L, \boldsymbol{\mu}, \kappa | \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}, \{(\mathbf{y}_1, \mathbf{u}_1), \ldots, (\mathbf{y}_k, \mathbf{u}_k)\}) \\
& = f(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} | \{(\mathbf{y}_1, \mathbf{u}_1), \ldots, (\mathbf{y}_k, \mathbf{u}_k)\}, \alpha, \sigma^2) f(\{(\mathbf{y}_1, \mathbf{u}_1), \ldots, (\mathbf{y}_k, \mathbf{u}_k)\} | \rho_L, \boldsymbol{\mu}, \kappa)
\end{aligned}
$$

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\end{aligned}
$$

- Thus the posterior density:

$$
\begin{aligned}
& p(\alpha, \sigma^2, \rho_L, \boldsymbol{\mu}, \kappa, \{(\mathbf{y}_1, \mathbf{u}_1), \dots, (\mathbf{y}_k, \mathbf{u}_k)\} | \{\mathbf{x}_1, \dots, \mathbf{x}_n\}) \\
& \propto l(\alpha, \sigma^2, \rho_L, \boldsymbol{\mu}, \kappa | \{\mathbf{x}_1, \dots, \mathbf{x}_n\}, \{(\mathbf{y}_1, \mathbf{u}_1), \dots, (\mathbf{y}_k, \mathbf{u}_k)\}) p(\alpha) p(\sigma^2) p(\rho_L) p(\boldsymbol{\mu}) p(\kappa).
\end{aligned}
$$

# A hybrid MCMC algorithm for estimating the parameters

$\alpha$ and $\rho_L$: Gibbs updates from two gamma distributions

$$R_{\sigma^2} = \frac{p(\sigma'^2)}{p(\sigma^2)} \exp\left(\alpha \int_W \sum_{j=1}^{k} \left[ f(p_{\mathbf{u}_j^\perp}(\mathbf{x}-\mathbf{y}_j)|\sigma^2) - f(p_{\mathbf{u}_j^\perp}(\mathbf{x}-\mathbf{y}_j)|\sigma'^2) \right] \mathrm{d}\mathbf{x} \right)$$

$$\prod_{i=1}^{n} \frac{\sum_{j=1}^{k} f(p_{\mathbf{u}_j^\perp}(\mathbf{x}_i-\mathbf{y}_j)|\sigma'^2)}{\sum_{j=1}^{k} f(p_{\mathbf{u}_j^\perp}(\mathbf{x}_i-\mathbf{y}_j)|\sigma^2)},$$

$$R_{\boldsymbol{\mu}} = \frac{p(\boldsymbol{\mu}')}{p(\boldsymbol{\mu})} \exp\left(\rho_L \left[ I(\boldsymbol{\mu},\kappa) - I(\boldsymbol{\mu}',\kappa) \right]\right) \prod_{j=1}^{k} \frac{f(\mathbf{u}_j|\boldsymbol{\mu}',\kappa)}{f(\mathbf{u}_j|\boldsymbol{\mu},\kappa)},$$

$$R_{\kappa} = \frac{p(\kappa')}{p(\kappa)} \exp\left(\rho_L \left[ I(\boldsymbol{\mu},\kappa) - I(\boldsymbol{\mu},\kappa') \right]\right) \prod_{j=1}^{k} \frac{f(\mathbf{u}_j|\boldsymbol{\mu},\kappa')}{f(\mathbf{u}_j|\boldsymbol{\mu},\kappa)}.$$

# Updating the missing data: The birth-death-move Metropolis-Hastings algorithm

$$R_{\text{birth}} = R_{\text{birth}}(k, \mathbf{y}, \mathbf{u}) = \frac{\rho_L \lambda(J_{\mathbf{u}})|u_d|}{k+1} \mathbf{1}[l(\mathbf{y}, \mathbf{u}) \cap W_{\text{ext}} \neq \emptyset]$$

$$R_{\text{death}} = R_{\text{death}}(k, \mathbf{y}_j, \mathbf{u}_j) = \frac{k}{\rho_L \lambda(J_{\mathbf{u}_j})|u_{d,j}|}$$

$$R_{\text{move}} = R_{\text{death}}(k, \mathbf{y}_j, \mathbf{u}_j) R_{\text{birth}}(k-1, \mathbf{y}'_j, \mathbf{u}'_j) = \frac{|u'_{d,j}|\lambda(J_{\mathbf{u}'_j})}{|u_{d,j}|\lambda(J_{\mathbf{u}_j})} \mathbf{1}[l(\mathbf{y}'_j, \mathbf{u}'_j) \cap W_{\text{ext}} \neq \emptyset]$$

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Thank you!