Half-Spectral Space-Time Covariance Models and a Natural Condition

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June 11, 2014

Half-Spectral Representations of Space-Time Covariance Functions

$$K(s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} g(\lambda,\omega) \exp\left(is^T\lambda + it\omega\right) d\lambda \; d\omega \tag{1}$$

$$K(s,t) = \int_{\mathbb{R}} f(\omega)\mathbb{C}(s\delta(\omega)) \exp(it\omega) d\omega, \tag{2}$$

Natural Condition in Stein (2011)

$$K(s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} g(\lambda, \omega) \exp\left(is^T \lambda + it\omega \right) d\lambda \; d\omega$$

$$\lim_{\|(\omega,\lambda)\| \to \infty} \sup_{\|(u,v)\| < R} \left| \frac{g(\omega + u, \lambda + v)}{g(\omega, \lambda)} - 1 \right| = 0 \qquad (3)$$

- Developed to ensure "nice" behavior of kriging predictors for models with g as a spectral density.

Goal

- Develop covariance functions that satisfy the condition in Stein (2011) that also have a half-spectral representation.

Spectral Form and Restrictions

$$K(s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(\omega)\delta(\omega)^{-d} h\left(\frac{\lambda}{\delta(\omega)}\right) \exp(it\omega + is^T\lambda) d\lambda d\omega.$$

- Restriction 1: If $h(\lambda/\delta(\omega)$ satisfies the condition in (3), $f(\omega)\delta(\omega)^{-d}$ must be constant.
- Restriction 2: Function $\delta(\omega)$ must increase unboundedly for large ω.
- A Conclusion:

$$h(\lambda/\delta(\omega)) = \delta(\omega)^d / f(\omega) H(\omega, \lambda)$$

where H satisfies the natural condition.

A New Class of Space-Time Models defined by Half-Spectrum

Let covariance function K_f have the following half-spectral representation

$$f(\omega)\mathbb{C}(s\delta(\omega)) = \phi f(\omega)\mathcal{M}_{\nu+1/2}\left(\alpha|s|f(\omega)^{-1/2\frac{1}{\nu+1/2}}\right).$$

- $\mathcal{M}_{\nu+1/2}$ is the Matérn covariance function with smoothness $\nu + 1/2$.
- Temporal spectrum f decreases to zero asymptotically proportional to ω^{-k}, where $k > 1$.

Example

Let f be a Matérn spectral density with smoothness κ.

$$g(\lambda, \omega) = \phi \left(\alpha^2 (\beta^2 + \omega^2)^{(\kappa+1/2)/(\nu+1/2)} + \lambda^2 \right)^{-(\nu+(d+1)/2)}$$

$$f(\omega)\mathbb{C}(s\delta(\omega)) = \phi \left(\beta^2 + \omega^2 \right)^{-(\kappa+1/2)} \mathcal{M}_{\nu+1/2} \left(\alpha |s| (\beta^2 + \omega^2)^{1/2 \frac{\kappa+1/2}{\nu+1/2}} \right)$$

- Temporal smoothness is κ.
- Spatial smoothness is $\kappa \frac{\nu+1/2}{\kappa+1/2}$

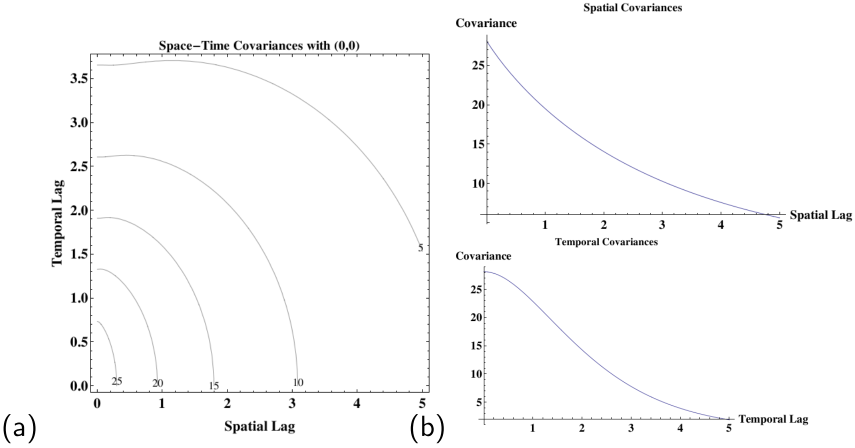

Figure: Contour and marginal covariance plots for the model in the example. Effective smoothness in space is 0.5. Effective smoothness in time is 2.

Application

- Irish wind dataset (Haslett and Raftery (1989))
- Daily average wind speeds at 11 stations
- 1961-1978
- 72,314 observations with none missing

Short Memory Model Comparisons

Whittle Log-Likelihoods of Short Memory Models

	Log-Likelihood	Diff. From Ex. 1	# Param. Fit
Example ($\kappa = 1/2$)	20,318	0	5
Matérn (Example, $\kappa = \nu = 1/2$)	20,199	119	4
Separable Exponential	18,703	1,615	4
Cressie and Huang (1999)	18,378	1,940	4

Table: Comparison of log-likelihoods of short memory models fitted to the Irish wind dataset.

- Note a spatial nugget (as seen on the following slide) is included in all these models.

Long Memory Model Comparisons

Gneiting (2002) analyzes these data using the following model.

$$G(s,t) = \phi(\beta|u|^{\kappa}+1)^{-1} \exp\left(-\frac{\alpha\|s\|}{(\beta|u|^{\kappa}+1)^{\gamma/2}}\right) + \eta^2(\beta|u|^{\kappa}+1)^{-1}\mathbf{1}_{s=0}$$

We fit versions of this model to the Irish wind dataset

- $\kappa = 1$ and $\gamma = 0$.
- $\kappa = 1$.

When $\kappa = 1$, let f_G be the spectrum of $G(0,t)$. We compare these versions of G to K_{f_G}.

Comparison to Separable G

Whittle Log-Likelihoods of Longer Memory Models

	Log-Likelihood	Difference From K_{f_G}	# Parameters Fit
K_{f_G} $(\nu = 1/2)$	21,655	0	4
G $(\gamma = 0,\ \kappa = 1)$	21,245	410	4

Table: Comparison of log-likelihoods of longer memory models fitted to the Irish wind dataset.

Comparison to Non-Separable G

Space-Time Log-Likelihoods of Long Memory Models

	Log-Likelihood	Difference From K_{f_G}	# Parameters Fit
K_{f_G} $(\nu = 1/2)$	6,744	0	4
G $(\kappa = 1)$	6,705	39	5

Table: Comparison of log-likelihoods of longer memory models fitted to a subset of the Irish wind dataset containing the first 2,000 temporal points. In contrast to Table 2, G is a non-separable model.

References

- Cressie, N. and Huang, H.-C. (1999). Classes of nonseparable, spatio-temporal stationary covariance functions. *J. Amer. Statist. Assoc.* **94**, 1330-1340.
- Gneiting, T. (2002). Nonseparable, stationary covariance functions for spacetime data. *J. Amer. Statist. Assoc.* **97**, 590-600.
- Haslett, J. and Raftery, A.E. (1989). Space-time modeling with long-memory dependence: Assesssing Ireland's wind power resource. *Appl. Stat.* **38** 1-50.
- Stein, M.L. (2011). 2010 Rietz Lecture: When does the screening effect hold? *Ann. Appl. Stat.*. **39**, 2795-2819.