Modelling Multivariate Spatial Data

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Bivariate Linear Spatial Regression

- A single covariate $X(\mathbf{s})$ and a univariate response $Y(\mathbf{s})$
- At any arbotrary point in the domain, we conceive a linear spatial relationship:
$$\mathrm{E}[Y(\mathbf{s}) \,|\, X(\mathbf{s})] = \beta_0 + \beta_1 X(\mathbf{s});$$
where $X(\mathbf{s})$ and $Y(\mathbf{s})$ are spatial processes.
- Regression on uncountable sets:
$$\text{Regress} \quad \{Y(\mathbf{s}) : \mathbf{s} \in \mathscr{D}\} \quad \text{on} \quad \{X(\mathbf{s}) : \mathbf{s} \in \mathscr{D}\}\ .$$
- Inference:
  - Estimate β_0 and β_1.
  - Estimate spatial surface $\{X(\mathbf{s}) : \mathbf{s} \in \mathscr{D}\}$.
  - Estimate spatial surface $\{Y(\mathbf{s}) : \mathbf{s} \in \mathscr{D}\}$.

Bivariate spatial process

- A bivariate distribution $[Y, X]$ will yield regression $[Y \mid X]$.
- So why not start with a bivariate process?

$$\mathbf{Z}(\mathbf{s}) = \begin{bmatrix} X(\mathbf{s}) \\ Y(\mathbf{s}) \end{bmatrix} \sim GP_2\left(\begin{bmatrix} \mu_X(\mathbf{s}) \\ \mu_Y(\mathbf{s}) \end{bmatrix}, \begin{bmatrix} C_{XX}(\cdot;\boldsymbol{\theta}_Z) & C_{XY}(\cdot;\boldsymbol{\theta}_Z) \\ C_{YX}(\cdot;\boldsymbol{\theta}_Z) & C_{YY}(\cdot;\boldsymbol{\theta}_Z) \end{bmatrix} \right)$$

- The cross-covariance function:

$$\mathbf{C}_Z(\mathbf{s},\mathbf{t};\boldsymbol{\theta}_Z) = \begin{bmatrix} C_{XX}(\mathbf{s},\mathbf{t};\boldsymbol{\theta}_Z) & C_{XY}(\mathbf{s},\mathbf{t};\boldsymbol{\theta}_Z) \\ C_{YX}(\mathbf{s},\mathbf{t};\boldsymbol{\theta}_Z) & C_{YY}(\mathbf{s},\mathbf{t};\boldsymbol{\theta}_Z) \end{bmatrix},$$

where $C_{XY}(\mathbf{s},\mathbf{t}) = \text{cov}(X(\mathbf{s}), Y(\mathbf{t}))$ and so on.

- Cross-covariance functions satisfy certain properties:

$$C_{XY}(\mathbf{s},\mathbf{t}) = \mathrm{cov}(X(\mathbf{s}), Y(\mathbf{t})) = \mathrm{cov}(Y(\mathbf{t}), X(\mathbf{s})) = C_{YX}(\mathbf{t},\mathbf{s}).$$

Caution: $C_{XY}(\mathbf{s},\mathbf{t}) \neq C_{XY}(\mathbf{t},\mathbf{s})$ and $C_{XY}(\mathbf{s},\mathbf{t}) \neq C_{YX}(\mathbf{s},\mathbf{t})$.

- In matrix terms, $\mathbf{C}_Z(\mathbf{s},\mathbf{t};\boldsymbol{\theta}_Z)^\top = \mathbf{C}_Z(\mathbf{t},\mathbf{s};\boldsymbol{\theta}_Z)$

- Positive-definiteness for any finite collection of points:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}_i^\top \mathbf{C}_Z(\mathbf{s}_i,\mathbf{t}_j;\boldsymbol{\theta}_Z)\mathbf{a}_j > 0 \;\text{ for all }\; \mathbf{a}_i \in \Re^2 \setminus \{\mathbf{0}\}.$$

Bivariare Spatial Regression from a Separable Process

- To ensure $\mathsf{E}[Y(\mathbf{s}) \,|\, X(\mathbf{s})] = \beta_0 + \beta_1 X(\mathbf{s})$, we must have

$$\mathbf{Z}(\mathbf{s}) = \begin{bmatrix} X(\mathbf{s}) \\ Y(\mathbf{s}) \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}\right) \text{ for every } \mathbf{s} \in \mathscr{D}$$

- Simplifying assumption :

$$\mathbf{C}_Z(\mathbf{s}, \mathbf{t}) = \rho(\mathbf{s}, \mathbf{t})\mathbf{T} \Longrightarrow \boldsymbol{\Sigma}_Z = \{\rho(\mathbf{s}_i, \mathbf{s}_j)\mathbf{T}\} = \mathbf{R}(\phi) \otimes \mathbf{T} .$$

- Then, $p(Y(\mathbf{s}) \mid X(\mathbf{s})) = N(Y(\mathbf{s}) \mid \beta_0 + \beta_1 X(\mathbf{s}), \sigma^2)$, where

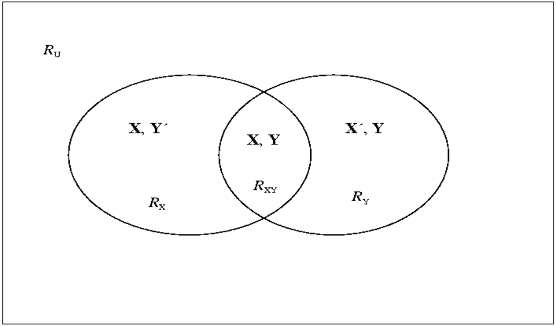
$$\beta_0 = \mu_2 - \frac{T_{12}}{T_{11}}\mu_1,$$
$$\beta_1 = \frac{T_{12}}{T_{11}},$$
$$\sigma^2 = T_{22} - \frac{T_{12}^2}{T_{11}} .$$

- Regression coefficients are functions of process parameters.
- Estimate $\{\mu_1, \mu_2, T_{11}, T_{12}, T_{22}\}$ by sampling from

$$p(\phi) \times N(\boldsymbol{\mu} \mid \boldsymbol{\delta}, \mathbf{V}_\mu) \times IW(\mathbf{T} \mid r, \mathbf{S}) \times N(\mathbf{Z} \mid \boldsymbol{\mu}, \mathbf{R}(\phi) \otimes \mathbf{T})$$

- Immediately obtain posterior samples of $\{\beta_0, \beta_1, \sigma^2\}$.

Misaligned Spatial Data



Bivariate Spatial Regression with Misalignment

- Rearrange the components of $\mathbf{Z}$ to $\widetilde{\mathbf{Z}} = (X(\mathbf{s}_1), X(\mathbf{s}_2), \ldots, X(\mathbf{s}_n), Y(\mathbf{s}_1), Y(\mathbf{s}_2), \ldots, Y(\mathbf{s}_n))^\top$ yields

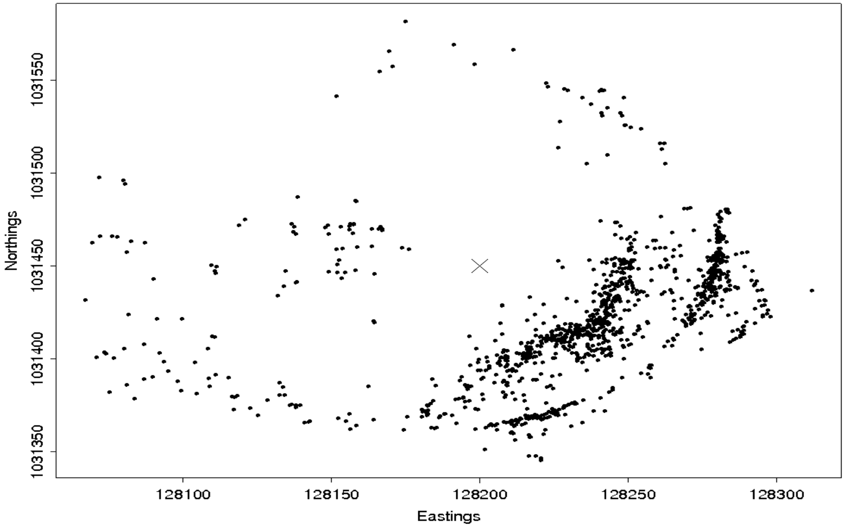
$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \mathbf{1} \\ \mu_2 \mathbf{1} \end{bmatrix}, \mathbf{T} \otimes \mathbf{R}(\phi) \right) .$$

- Priors: Wishart for T^{-1}, normal (perhaps flat) for (μ_1, μ_2), discrete prior for ϕ or perhaps a uniform on $(0, .5\text{max dist})$.

- Estimation: Markov chain Monte Carlo (Gibbs, Metropolis, Slice, HMC/NUTS); Integrated Nested Laplace Approximation (INLA).

Dew-Shrub Data from Negev Desert in Israel

- Negev desert is very arid
- Condensation can contribute to annual water levels
- Analysis: Determine impact of shrub density on dew duration
- 1129 locations with UTM coordinates
- $X(\mathbf{s})$: Shrub density at location **s** (within $5m \times 5m$ blocks)
- $Y(\mathbf{s})$: Dew duration at location **s** (in 100-th of an hour)
- Separable model with an exponential correlation function, $\rho(\|\mathbf{s}-\mathbf{t}\|;\phi) = e^{-\phi\|\mathbf{s}-\mathbf{t}\|}$


Northings
1031350
1031400
1031450
1031500
1031550
128100
128150
128200
128250
128300
Eastings

Parameter	2.5%	50%	97.5%
μ_1	73.12	73.89	74.67
μ_2	5.20	5.38	5.572
T_{11}	95.10	105.22	117.69
T_{12}	–4.46	–2.42	–0.53
T_{22}	5.56	6.19	6.91
ϕ	0.01	0.03	0.21
β_0	5.72	7.08	8.46
β_1	–0.04	–0.02	–0.01
σ^2	5.58	6.22	6.93
$T_{12}/\sqrt{T_{11}T_{22}}$	–0.17	–0.10	–0.02

Hierarchical approach (Royle and Berliner, 1999; Cressie and Wikle, 2011)

- $Y(\mathbf{s})$ and $X(\mathbf{s})$ observed over a finite set of locations $\mathscr{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$.
- $\mathbf{Y}$ and $\mathbf{X}$ are $n \times 1$ vectors of observed $Y(\mathbf{s}_i)$'s and $X(\mathbf{s}_i)$'s, respectively.
- How do we model $\mathbf{Y} \mid \mathbf{X}$?
- No "conditional process"—meaningless to talk about the joint distribution of $Y(\mathbf{s}_i) \mid X(\mathbf{s}_i)$ and $Y(\mathbf{s}_j) \mid X(\mathbf{s}_j)$ for two distinct locations $\mathbf{s}_i$ and $\mathbf{s}_j$.
- Can model using $[\mathbf{X}] \times [\mathbf{Y} \mid \mathbf{X}]$ but can we interpolate/predict at arbitrary locations?

Hierarchical approach (contd.)

- $X(\mathbf{s}) \sim GP(\mu_X(\mathbf{s}), C_X(\cdot; \boldsymbol{\theta}_X))$. Therefore, $\mathbf{X} \sim N(\boldsymbol{\mu}_X, \mathbf{C}_X(\boldsymbol{\theta}_X))$.
- $\mathbf{C}_X(\boldsymbol{\theta}_X)$ is $n \times n$ with entries $C_X(\mathbf{s}_i, \mathbf{s}_j; \boldsymbol{\theta}_X)$.
- $e(\mathbf{s}) \sim GP(0, C_e(\cdot; \boldsymbol{\theta}_e))$; $\mathbf{C}_e$ is analogous to $\mathbf{C}_X$.

$$Y(\mathbf{s}_i) = \beta_0 + \beta_1 X(\mathbf{s}_i) + e(\mathbf{s}_i)\ , \quad \text{for} \ \ i = 1, 2, \ldots, n\ .$$

- Joint distribution of $\mathbf{Y}$ and $\mathbf{X}$:

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_X \\ \boldsymbol{\mu}_Y \end{bmatrix}, \begin{bmatrix} \mathbf{C}_X(\boldsymbol{\theta}_X) & \beta_1 \mathbf{C}_X(\boldsymbol{\theta}_X) \\ \beta_1 \mathbf{C}_X(\boldsymbol{\theta}_X) & \mathbf{C}_e(\boldsymbol{\theta}_e) + \beta_1^2 \mathbf{C}_X(\boldsymbol{\theta}_X) \end{bmatrix} \right),$$

where $\boldsymbol{\mu}_Y = \beta_0 \mathbf{1} + \beta_1 \boldsymbol{\mu}_X$.

This joint distribution arises from a bivariate spatial process:

$$\mathbf{W}(\mathbf{s}) = \begin{bmatrix} X(\mathbf{s}) \\ Y(\mathbf{s}) \end{bmatrix} \quad \text{and} \quad \mathrm{E}[\mathbf{W}(\mathbf{s})] = \boldsymbol{\mu}_W(\mathbf{s}) = \begin{bmatrix} \mu_X(\mathbf{s}) \\ \beta_0 + \beta_1 \mu_X(\mathbf{s}) \end{bmatrix} .$$

and cross-covariance

$$C_{\mathbf{W}}(\mathbf{s}, \mathbf{s}') = \begin{bmatrix} C_X(\mathbf{s}, \mathbf{s}') & \beta_1 C_X(\mathbf{s}, \mathbf{s}') \\ \beta_1 C_X(\mathbf{s}, \mathbf{s}') & \beta_1^2 C_X(\mathbf{s}, \mathbf{s}') + C_e(\mathbf{s}, \mathbf{s}') \end{bmatrix} ,$$

where we have suppressed the dependence of $C_X(\mathbf{s}, \mathbf{s}')$ and $C_e(\mathbf{s}, \mathbf{s}')$ on $\boldsymbol{\theta}_X$ and $\boldsymbol{\theta}_e$ respectively. This implies that $\mathrm{E}[Y(\mathbf{s}) \,|\, X(\mathbf{s})] = \beta_0 + \beta_1 X(\mathbf{s})$ for any arbitrary location $\mathbf{s}$, thereby specifying a well-defined spatial regression model for an arbitrary $\mathbf{s}$.

Coregionalization (Wackernagel)

- Separable models assume one spatial range for both $X(\mathbf{s})$ and $Y(\mathbf{s})$.
- Coregionalization helps to introduce a second "range parameter."
- Introduce two "latent" independent GP's, each having its own parameters:

$$v_1(\mathbf{s}) \sim GP(0, \rho_1(\cdot;\phi_1)) \quad \text{and} \quad v_2(\mathbf{s}) \sim GP(0, \rho_2(\cdot;\phi_2))$$

- Construct a bivariate process as the linear transformation:

$$\begin{aligned} w_1(\mathbf{s}) &= a_{11}v_1(\mathbf{s}) \\ w_2(\mathbf{s}) &= a_{21}v_1(\mathbf{s}) + a_{22}v_2(\mathbf{s}) \end{aligned}$$

Coregionalization

- Short form:

$$\mathbf{w}(\mathbf{s}) = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1(\mathbf{s}) \\ v_2(\mathbf{s}) \end{bmatrix} = \mathbf{A}\mathbf{v}(\mathbf{s})$$

- Cross-covariance of $\mathbf{v}(\mathbf{s})$:

$$\mathbf{C}_v(\mathbf{s}, \mathbf{t}) = \begin{bmatrix} \rho_1(\mathbf{s}, \mathbf{t}; \phi_1) & 0 \\ 0 & \rho_2(\mathbf{s}, \mathbf{t}; \phi_2) \end{bmatrix}$$

- Cross-covariance of $\mathbf{w}(\mathbf{s})$:

$$\mathbf{C}_w(\mathbf{s}, \mathbf{t}) = \mathbf{A}\mathbf{C}_v(\mathbf{s}, \mathbf{t})\mathbf{A}^\top .$$

  It is a valid cross-covariance function (by construction).

- If $\mathbf{s} = \mathbf{t}$, then $\mathbf{C}_w(\mathbf{s}, \mathbf{s}) = \mathbf{A}\mathbf{A}^\top$. No loss of generality to speficy $\mathbf{A}$ as (lower) triangular.

- If $v_1(\mathbf{s})$ and $v_2(\mathbf{s})$ have identical correlation functions, then $\rho_1(\mathbf{s},\mathbf{t}) = \rho_2(\mathbf{s},\mathbf{t})$ and

$$\mathbf{C}_w(\mathbf{s}) = \rho(\mathbf{s},\mathbf{t};\phi)\mathbf{A}\mathbf{A}^\top \implies \text{separable model}$$

- Coregionalized Spatial Linear Model

$$\begin{bmatrix} X(\mathbf{s}) \\ Y(\mathbf{s}) \end{bmatrix} = \begin{bmatrix} \mu_X(\mathbf{s}) \\ \mu_Y(\mathbf{s}) \end{bmatrix} + \begin{bmatrix} w_1(\mathbf{s}) \\ w_2(\mathbf{s}) \end{bmatrix} + \begin{bmatrix} e_X(\mathbf{s}) \\ e_Y(\mathbf{s}) \end{bmatrix} ,$$

where $e_X(\mathbf{s})$ and $e_Y(\mathbf{s})$ are independent white-noise processes

$$\begin{bmatrix} e_X(\mathbf{s}) \\ e_Y(\mathbf{s}) \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_X^2 & 0 \\ 0 & \tau_Y^2 \end{bmatrix}\right) \text{ for every } \mathbf{s} \in \mathscr{D} .$$

Generalizations

- Each location contains m spatial regressions

$$Y_k(\mathbf{s}) = \mu_k(\mathbf{s}) + w_k(\mathbf{s}) + \epsilon_k(\mathbf{s}),\ k = 1, \ldots, m.$$

- Let $v_k(\mathbf{s}) \sim GP(0, \rho_k(\mathbf{s}, \mathbf{s}'))$, for $k = 1, \ldots, m$ be m independent GP's with unit variance.

- Assume $\mathbf{w}(\mathbf{s}) = A(\mathbf{s})\mathbf{v}(\mathbf{s})$ arises as a *space-varying* linear transformation of $\mathbf{v}(\mathbf{s})$. Then:

$$\mathbf{C}_w(\mathbf{s}, \mathbf{t}) = A(\mathbf{s})\mathbf{C}_v(\mathbf{s}, \mathbf{t})A^\top(\mathbf{t})$$

  is a valid cross-covariance function.

- $A(\mathbf{s})$ is unknown!
  - Should we first model $A(\mathbf{s})$ to obtain $\Gamma_{\mathbf{w}}(\mathbf{s}, \mathbf{s})$?
  - Or should we model $\Gamma_{\mathbf{w}}(\mathbf{s}, \mathbf{s}')$ first and derive $A(\mathbf{s})$?
  - $A(\mathbf{s})$ is completely determined from within-site associations.

- Moving average or kernel convolution of a process:
  - Let $Z(\mathbf{s}) \sim GP(0, \rho(\cdot))$. Use kernels to form:

$$w_j(\mathbf{s}) = \int \kappa_j(\mathbf{u}) Z(\mathbf{s}+\mathbf{u}) d\mathbf{u} = \int \kappa_j(\mathbf{s}-\mathbf{s}') Z(\mathbf{s}') d\mathbf{s}'$$

  - $\Gamma_{\mathbf{w}}(\mathbf{s}-\mathbf{s}')$ has (i,j)-th element:

$$[\Gamma_{\mathbf{w}}(\mathbf{s}-\mathbf{s}')]_{i,j} = \int\int \kappa_i(\mathbf{s}-\mathbf{s}'+\mathbf{u}) \kappa_j(\mathbf{u}') \rho(\mathbf{u}-\mathbf{u}') d\mathbf{u} d\mathbf{u}'$$

- Convolution of Covariance Functions:
  - $\rho_1, \rho_2, \ldots \rho_m$ are valid covariance functions. Form:

$$[\Gamma_{\mathbf{w}}(\mathbf{s}-\mathbf{s}')]_{i,j} = \int \rho_i(\mathbf{s}-\mathbf{s}'-\mathbf{t}) \rho_j(\mathbf{t}) d\mathbf{t}.$$

Other approaches for cross-covariance models

- Latent dimension approach:
  - Apanasovich and Genton (*Biometrika*, 2010).
  - Apanasovich et al. (*JASA*, 2012).
- Multivariate Matérn family
  - Gneiting et al. (*JASA*, 2010).
- Nonstationary variants of Coregionalization
  - Space-varying: Gelfand et al. (*Test*, 2010); Guhaniyogi et al. (*JABES*, 2012).
  - Multi-resolution: Banerjee and Johnson (*Biometrics*, 2006).
  - Dimension-reducing: Ren and Banerjee (*Biometrics*, 2013).
  - Variogram modeling: De Iaco et al. (*Math. Geo.*, 2003).

Big Multivariate Spatial Data

- Covariance tapering (Furrer et al. 2006; Zhang and Du, 2007; Du et al. 2009; Kaufman et al., 2009).
- Approximations using GMRFs: `INLA` (Rue et al. 2009; Lindgren et al., 2011).
- Nearest-neighbor models (processes) (Vecchia 1988; Stein et al. 2004; Datta et al., 2014)
- Low-rank approaches (Wahba, 1990; Higdon, 2002; Lin et al., 2000; Kamman & Wand, 2003; Paciorek, 2007; Rasmussen & Williams, 2006; Stein 2007, 2008; Cressie & Johannesson, 2008; Banerjee et al., 2008; 2010; Sang et al., 2011; Ren and Banerjee, 2013).

Illustration from:

Finley, A.O., S. Banerjee, A.R. Ek, and R.E. McRoberts. (2008) Bayesian multivariate process modeling for prediction of forest attributes. *Journal of Agricultural, Biological, and Environmental Statistics*, 13:60–83.

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- Evaluate methods for multi-source forest attribute mapping
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Study area:

- USDA FS Bartlett Experimental Forest (BEF), NH
- 1,053 ha heavily forested
- Major tree species: American beech (BE), eastern hemlock (EH), red maple (RM), sugar maple (SM), and yellow birch (YB)

Bartlett Experimental Forest

Image provided by `www.fs.fed.us/ne/durham/4155/bartlett`

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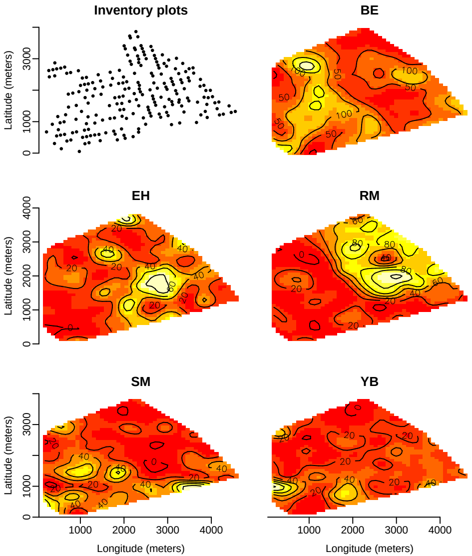
- Major tree species: American beech (BE), eastern hemlock (EH), red maple (RM), sugar maple (SM), and yellow birch (YB)

Response variables:

- Metric tons of total tree biomass per ha
- Measured on 437 $\frac{1}{10}$ ha plots
- Models fit using random subset of 218 plots
- Prediction at remaining 219 plots



Coregionalized model with dependence within locations

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Focus on spatial cross-covariance matrix $\mathbf{A}\mathbf{A}^\top$ (for brevity).

Posterior inference of `corr(AA`$^\top$`)`, e.g., 50 (2.5, 97.5) percentiles:

	BE	EH	...
BE	1		...
EH	0.16 (0.13, 0.21)	1	...
RM	-0.20 (-0.23, -0.15)	0.45 (0.26, 0.66)	...
SM	-0.20 (-0.22, -0.17)	-0.12 (-0.16, -0.09)	...
YB	0.07 (0.04, 0.08)	0.22 (0.20, 0.25)	...

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YB	0.07 (0.04, 0.08)	0.22 (0.20, 0.25)	...

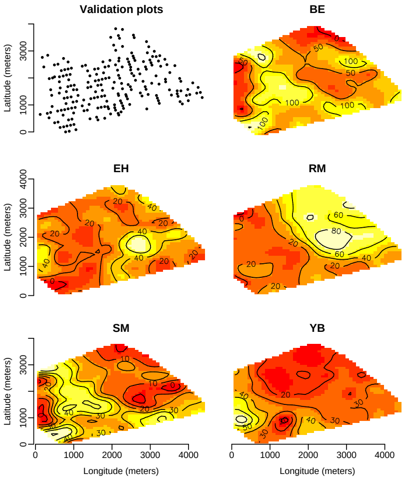
Parameters	50% (2.5%, 97.5%)
BE Model	
Intercept	-480.75 (-747.52, -213.54)
ELEV	0.17 (0.09, 0.24)
AprTC2	1.72 (0.69, 2.74)
AprTC3	-1.00 (-1.93, -0.06)
AugTC1	3.39 (2.25, 4.51)
AugTC3	1.45 (-0.25, 3.19)
OctTC2	-0.77 (-1.83, 0.25)
EH Model	
Intercept	-170.85 (-364.6, 21.45)
SLOPE	-0.95 (-1.69, -0.17)
AprTC1	2.08 (0.54, 3.66)
AprTC2	-0.87 (-1.85, 0.13)
AprTC3	1.75 (0.38, 3.13)
AugTC2	-0.65 (-1.11, -0.18)
AugTC3	1.54 (0.60, 2.51)
OctTC1	-1.74 (-2.76, -0.73)
OctTC2	1.55 (0.63, 2.45)
OctTC3	-1.27 (-2.24, -0.31)

Parameters	50% (2.5%, 97.5%)
RM Model	
Intercept	158.94 (16.92, 295.91)
ELEV	-0.07 (-0.14, 0.00)
SLOPE	-1.76 (-2.75, -0.77)
AprTC2	-0.87 (-1.42, -0.30)
AugTC3	1.30 (0.43, 2.14)
OctTC2	-0.95 (-1.61, -0.29)
SM Model	
Intercept	-97.71 (-191.86, 0.62)
SLOPE	1.11 (0.48, 1.74)
AugTC2	1.05 (0.71, 1.37)
AugTC3	-0.44 (-1.1, 0.21)
YB Model	
Intercept	-174.62 (-308.22, -29.63)
ELEV	0.08 (0.01, 0.13)
SLOPE	0.01 (-0.76, 0.82)
AugTC1	0.27 (-0.26, 0.77)
AugTC3	1.37 (0.47, 2.21)

Response	Estimates: 50% (2.5%, 97.5%)
BE	270.84 (200.15, 334.52)
EH	697.47 (466.23, 998.97)
RM	756.50 (504.08, 954.28)
SM	275.10 (207.13, 395.49)
YB	314.03 (253.71, 856.79)

Table: Distance in meters at which the spatial correlation drops to 0.05 for each of the response variables. Distance calculated by solving the Matérn correlation function for d using $\rho = 0.05$ and response specific ϕ and ν parameters estimates from co-regionalized model.

$$E[\mathbf{Y}^* \mid Data]$$



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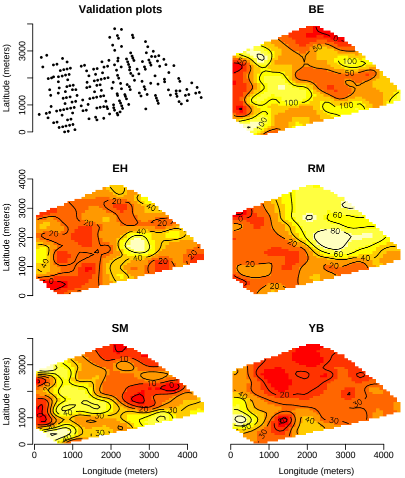
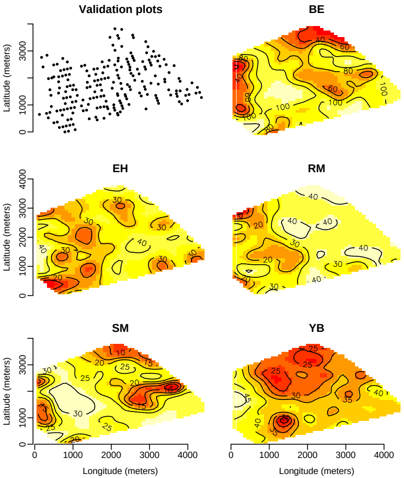


IQR for $\mathbf{Y}^*$



Available software for modeling and visualization

- `spBayes` `R` package for Bayesian hierarchical modeling for univariate (`spLM`) and multivariate (`spMvLM`) spatial data
  - `http://blue.for.msu.edu/software.html`
  - `http://cran.r-project.org`
- `MBA` `R` package for surface approximation/interpolation with multilevel B-splines
  - `http://blue.for.msu.edu/software.html`
  - `http://cran.r-project.org`

Manuscripts and course notes at `http://www.biostat.umn.edu/∼sudiptob` Also check out Andrew Finley's website: `http://blue.for.msu.edu/index.html`

Thank You!