Pan-American
Advanced Study Institute
on Spatio-Temporal Statistics

Purpose

Funding

Structure

Lectures

Practica

Applications

Workshops

Web page

main

data

Participant locations

Canada 4

US 33

Central America&Caribbean&Mexico 4

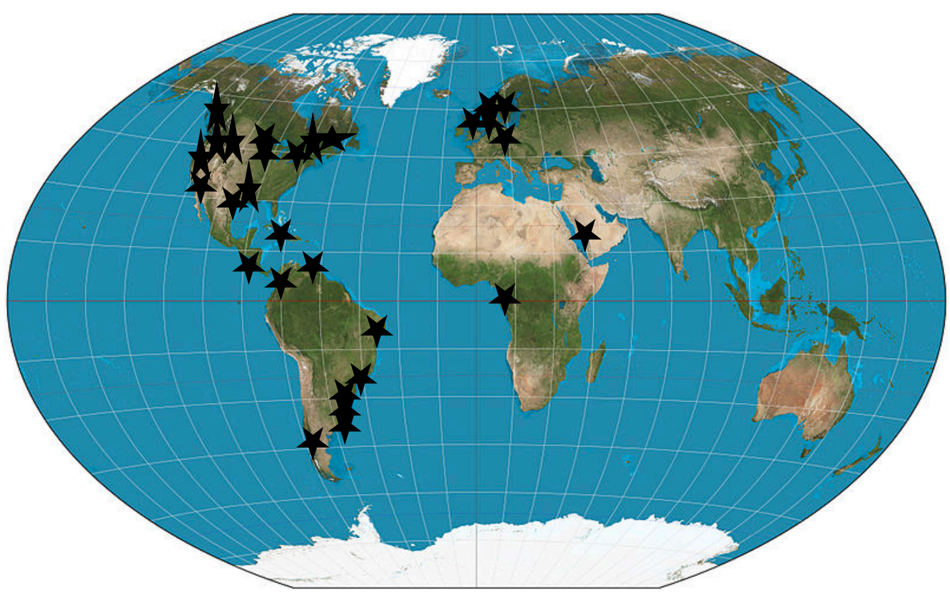
Brazil 15

South America exc. Brazil 10

Nordic 11

Other 7

Total 84



Kriging

The geostatistical model

Gaussian process $Z(s), s \in D \subseteq R^2$

$\mu(s)=EZ(s)$ Var $Z(s) < \infty$

Z is strictly stationary if

$$(Z(s_1),\ldots,Z(s_k)) \stackrel{d}{=} (Z(s_1+h),\ldots,Z(s_k+h))$$

Z is weakly stationary if

$$\mu(s) \equiv \mu \quad \mathrm{Cov}(Z(s_1),Z(s_2)) = C(s_1 - s_2)$$

Z is isotropic if weakly stationary and

$$C(s_1 - s_2) = C_0(\|s_1 - s_2\|)$$

The problem

Given observations at n locations

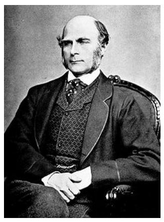
$$Z(s_1),\ldots,Z(s_n)$$

estimate

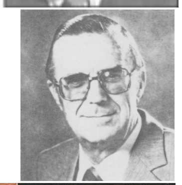
$Z(s_0)$ **(the process at an unobserved location)**

or $\int_A Z(s)d\nu(s)$ **(an average of the process)**

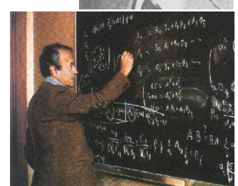
In the environmental context often time series of observations at the locations.

Some history

- Regression (Bravais, Galton, Bartlett)



- Mining engineers (Krige 1951, Matheron, 60s)

- Spatial models (Whittle, 1954)
- Forestry (Matérn, 1960)


- Objective analysis (Gandin, 1961)

- More recent work Cressie (1993), Stein (1999)
- Gelfand et al. (2010)

A Gaussian formula

If $$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}\right)$$

then $$(Y \mid X) \sim N(\mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(X - \mu_X),$$
$$\Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY})$$

Simple kriging

Let $X = (Z(s_1),\dots,Z(s_n))^T$, $Y = Z(s_0)$, so that

$\mu_X=\mu 1_n$, $\mu_Y=\mu$,

$\Sigma_{XX}=[C(s_i-s_j)]$, $\Sigma_{YY}=C(0)$, and

$\Sigma_{YX}=[C(s_i-s_0)]$.

Then

$$p(X) \equiv \hat{Z}(s_0) = \mu + \left[C(s_i - s_0)\right]^T \left[C(s_i - s_j)\right]^{-1} \left(X - \mu 1_n\right)$$

This is the best unbiased linear predictor when μ and C are known (simple kriging).

The prediction variance is

$$m_1 = C(0) - \left[C(s_i - s_0)\right]^T \left[C(s_i - s_j)\right]^{-1} \left[C(s_i - s_0)\right]$$

Some variants

Ordinary kriging (unknown μ)

$$p(X) \equiv \hat{Z}(s_0) = \hat{\mu} + \left[C(s_i - s_0)\right]^T \left[C(s_i - s_j)\right]^{-1} \left(X - \hat{\mu} 1_n\right)$$

where

$$\hat{\mu} = \left(1_n^T \left[C(s_i - s_j)\right]^{-1} 1_n\right)^{-1} 1_n^T \left[C(s_i - s_j)\right]^{-1} X$$

Universal kriging ($\mu(s)=A(s)\beta$ for some spatial variable A)

$$\hat{\beta} = (\left[A(s_i)\right]^T \left[C(s_i - s_j)\right]^{-1} \left[A(s_i)\right])^{-1}$$

$$\left[A(s_i)\right]^T \left[C(s_i - s_j)\right]^{-1} X$$

$$p(X) \equiv \hat{Z}(s_0) = \hat{\mu}(s_0) + \left[C(s_i - s_0)\right]^T \left[C(s_i - s_j)\right]^{-1} \left(X - [\hat{\mu}(s_i)]\right)$$

where $\hat{\mu}(s) = A(s)\hat{\beta}$

Still optimal for known C.

Universal kriging variance

simple kriging variance

$$E\left(\hat{Z}(s_0) - Z(s_0)\right)^2 = \boxed{m_1} +$$

$$\left(A(s_0) - [A(s_i)^T[C(s_i - s_j)]^{-1}[C(s_i - s_0)]\right)^T$$
$$\times\left(\left[A(s_i)\right]^T\left[C(s_i - s_j)\right]^{-1}\left[A(s_i)\right]\right)^{-1}$$
$$\times\left(A(s_0) - [A(s_i)^T[C(s_i - s_j)]^{-1}[C(s_i - s_0)]\right)$$

variability due to estimating μ

The (semi)variogram

$$\gamma(\|h\|) = \frac{1}{2}\mathrm{Var}(Z(s+h) - Z(s)) = C(0) - C(\|h\|)$$

Intrinsic stationarity

Weaker assumption (C(0) needs not exist)

Kriging predictions can be expressed in terms of the variogram instead of the covariance.

The exponential variogram

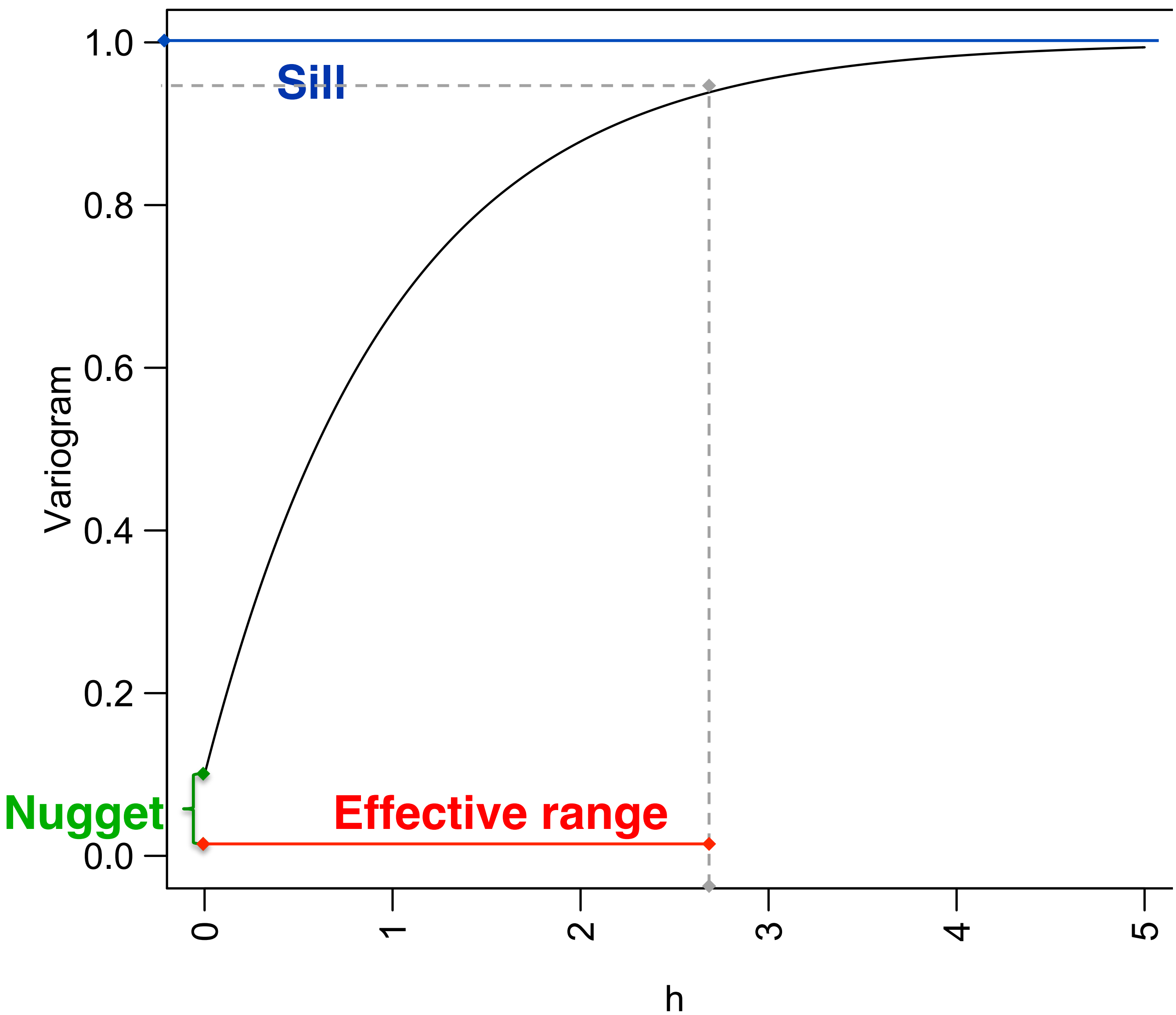
A commonly used variogram function is $\gamma(h) = \sigma^2 (1 - e^{-h/\varphi})$.

Corresponds to a Gaussian process with continuous but not differentiable sample paths.

More generally,

$$\gamma(h) = (\sigma^2 - \tau^2)\left(1 - e^{-h/\varphi}\right) + \tau^2$$

has a nugget τ^2, corresponding to measurement error and spatial correlation at small distances.



1.0
0.8
0.6
0.4
0.2
0.0
Variogram
Sill
Nugget
Effective range
0
1
2
3
4
5
h

Ordinary kriging

$$\hat{Z}(s_0) = \sum_{i=1}^{n} \lambda_i Z(s_i)$$

where

$$\lambda^T = \left(\gamma + \mathbf{1} \frac{1 - \mathbf{1}^T \Gamma^{-1} \gamma}{\mathbf{1}^T \Gamma^{-1} \mathbf{1}} \right)^T \Gamma^{-1}$$

$$\gamma = (\gamma(s_0 - s_1), \ldots, \gamma(s_0 - s_n))^T$$

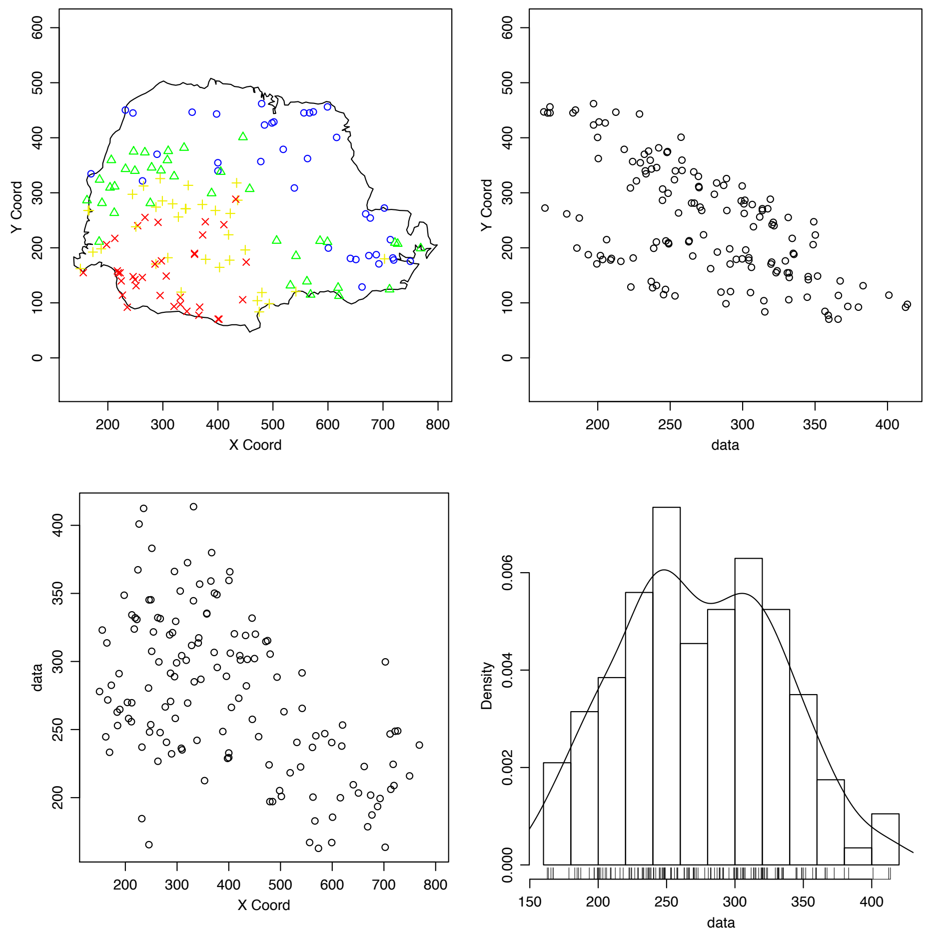
$$\Gamma_{ij} = \gamma(s_i - s_j)$$

and kriging variance

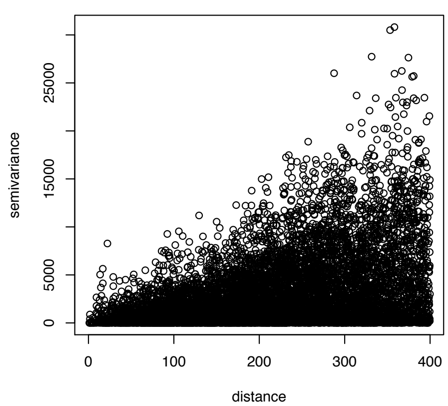
$$m_1(s_0) = 2\sum_{i=1}^{n} \lambda_i \gamma(s_0 - s_i) - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j \gamma(s_i - s_j)$$

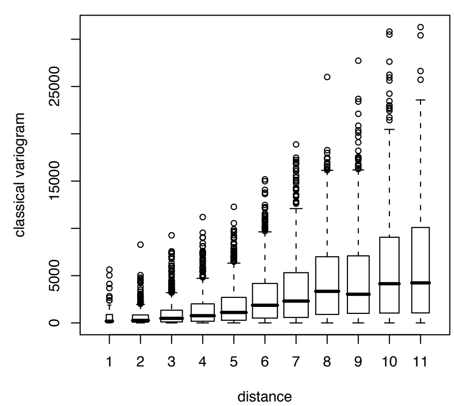
An example

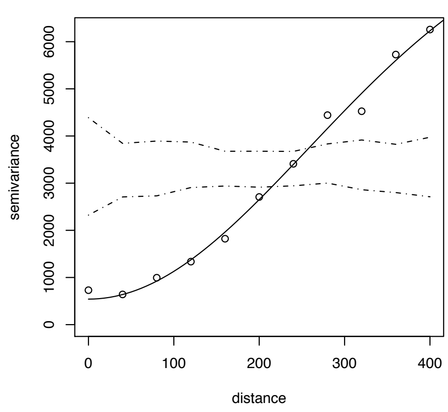
Precipitation data from Parana state in Brazil (May-June, averaged over years)



Variogram plots

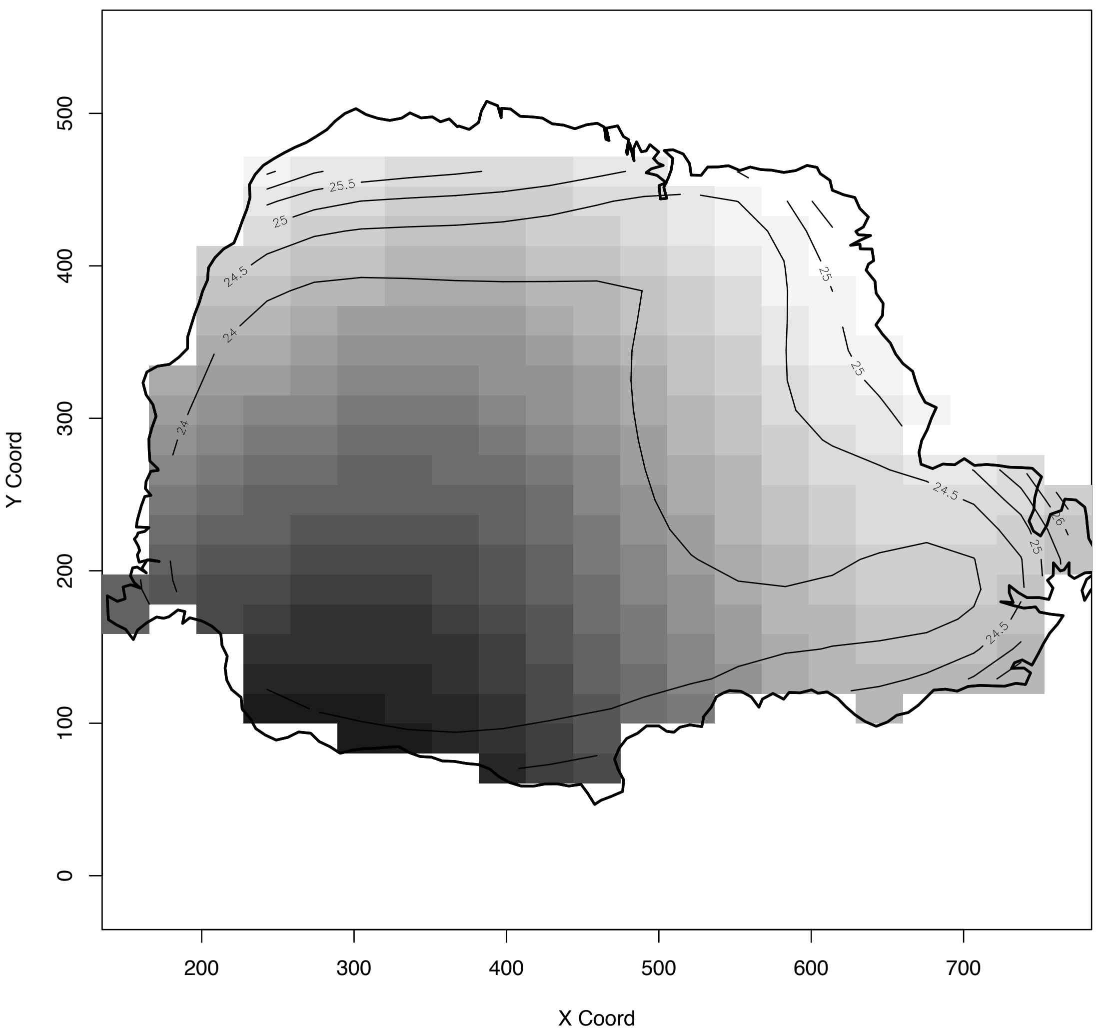






$$\gamma(h) = 542 + 8141\left(1 - \exp(-(\frac{h}{365})^2\right)$$

Kriging surface



Bayesian kriging

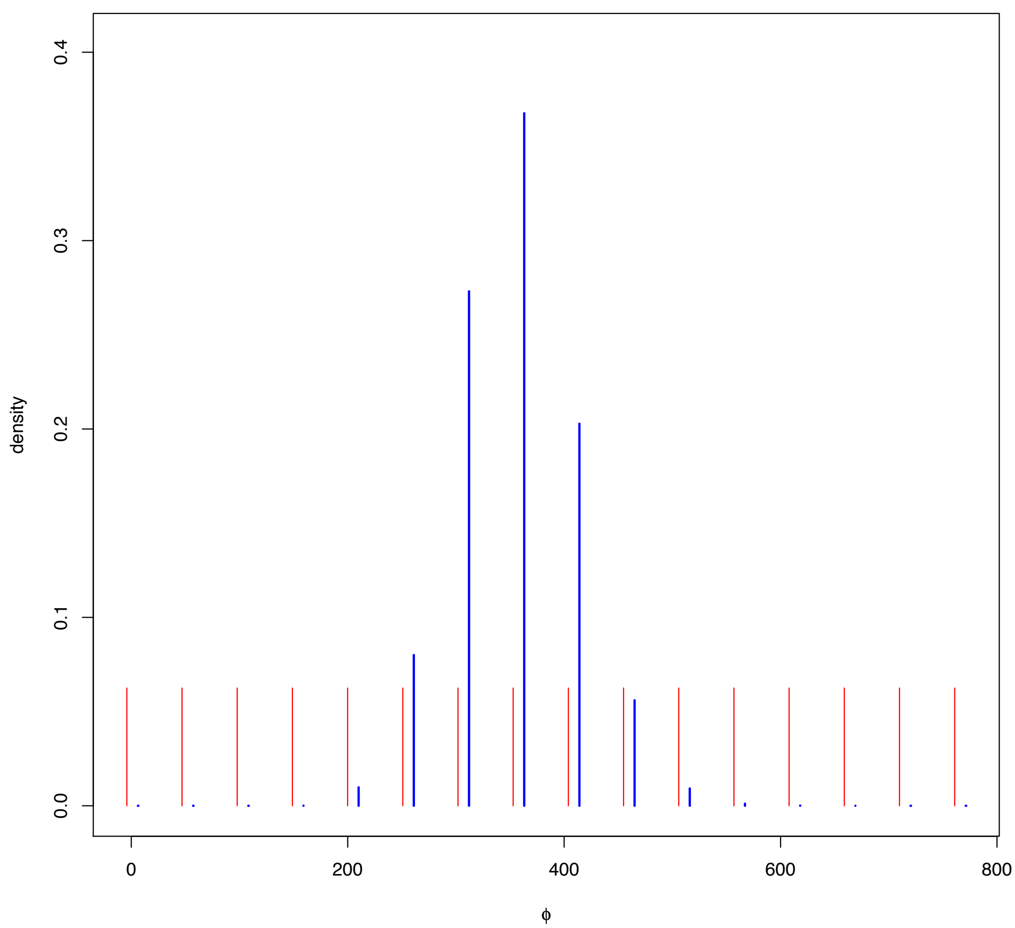
Instead of estimating the parameters, we put a prior distribution on them, and update the distribution using the data.

Model:

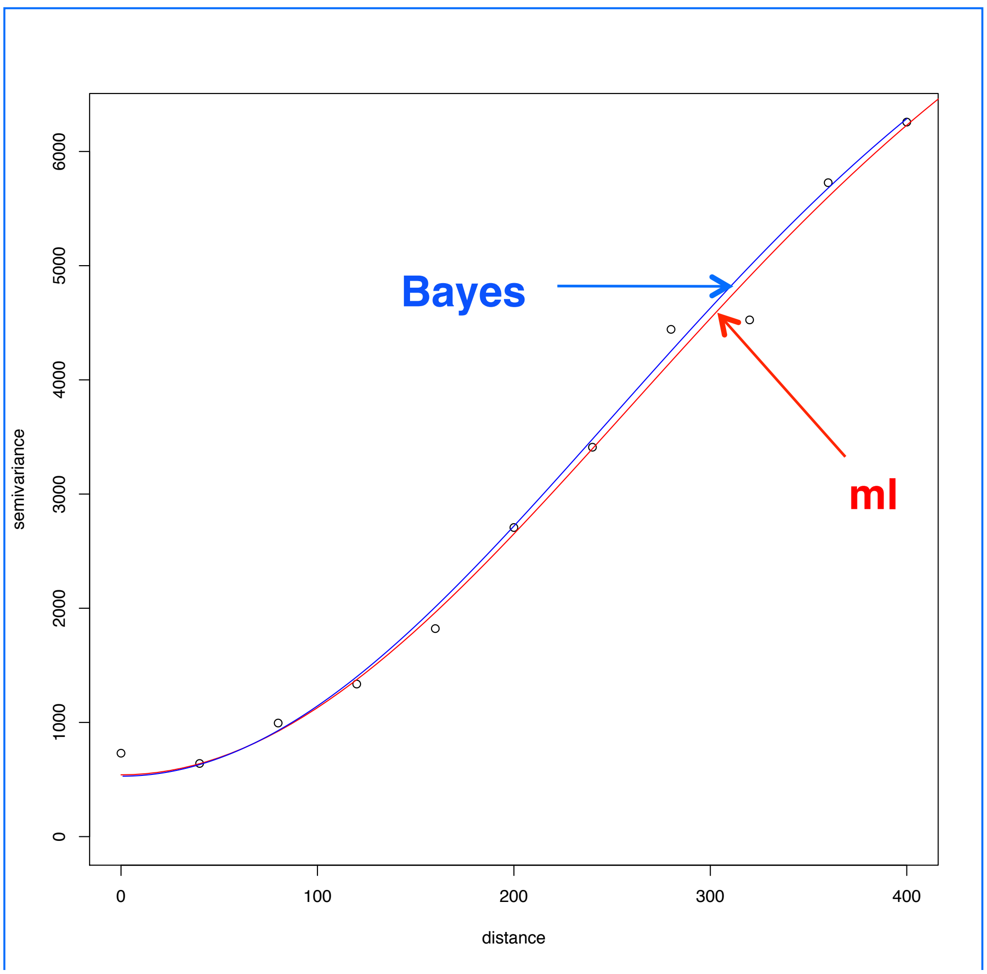
$$(\mathbf{X} \mid \theta) \sim N(\beta, \sigma^2 \mathbf{C}(\phi) + \tau^2 \mathbf{I})$$

- $\mathbf{X}$: $(Z(s_1)\ldots Z(s_n))^T$
- θ: $\theta=(\beta,\sigma^2,\phi,\tau^2)^T$
- $\mathbf{C}(\phi)$: **Matrix with i,j-element $C(s_i - s_j; \phi)$ (correlation)**
- $\tau^2 \mathbf{I}$: **measurement error**

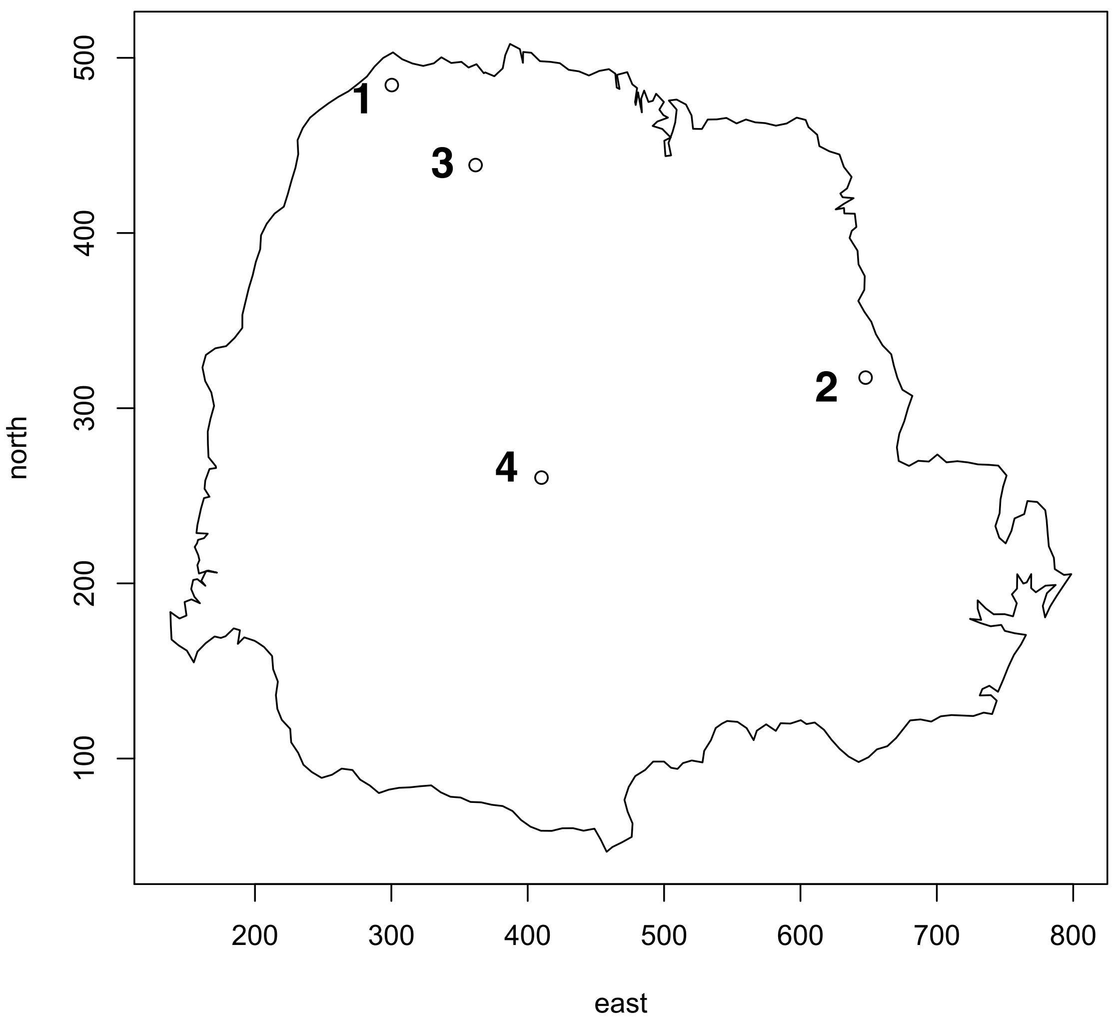
Prior/posterior of ϕ



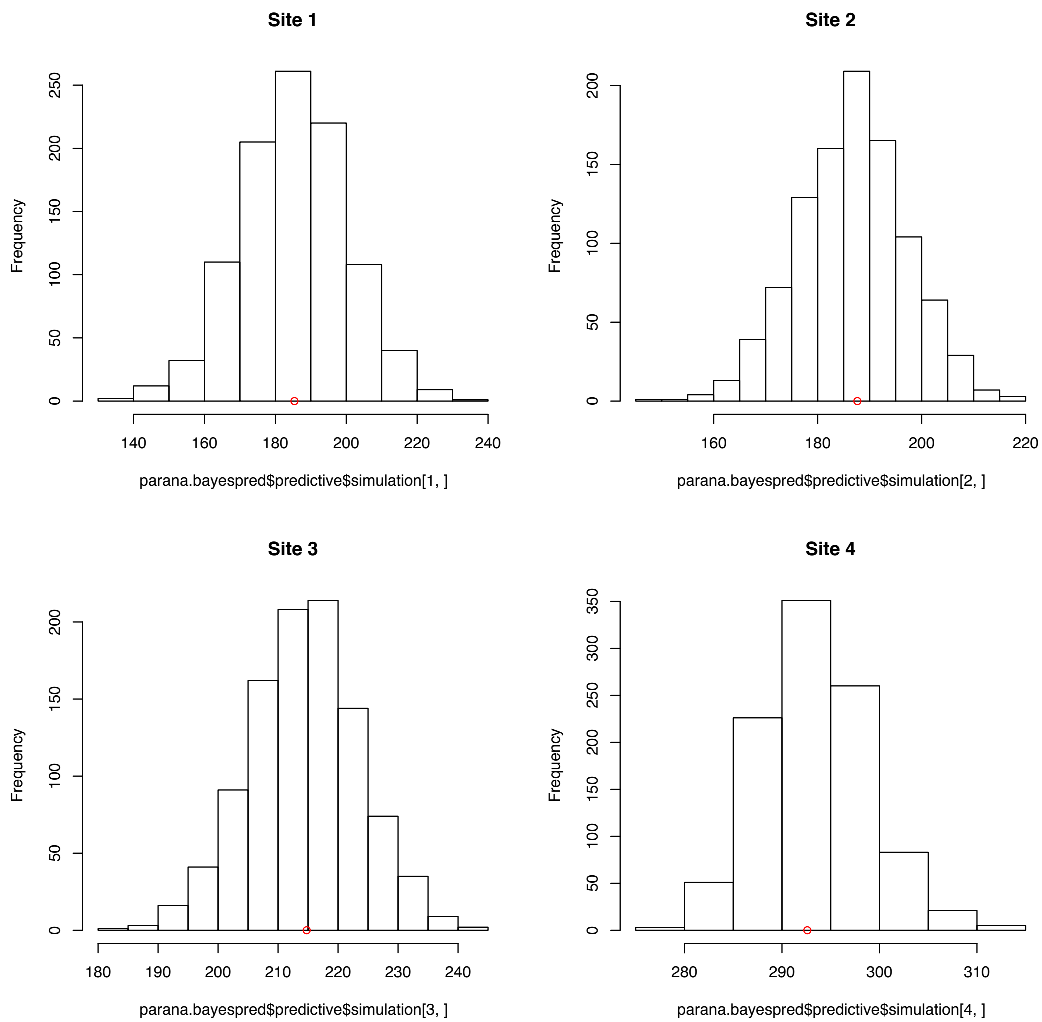
Estimated variogram



Prediction sites



Predictive distribution



References

A. Gelfand, P. Diggle, M. Fuentes and P. Guttorp, eds. (2010): *Handbook of Spatial Statistics*. Section 2, Continuous Spatial Variation. Chapman & Hall/CRC Press.

P.J. Diggle and Paulo Justiniano Ribeiro (2010): *Model-based Geostatistics*. Springer.