RandomFields V 3.0

Martin Schlather

Institute for Mathematics
Universität Mannheim

Joint work with
Daphne Boecker, Sebastian Engelke, Sebastian Gross, Alexander Malinowski, J. Martini, Peter Menck, Ulrike Ober, Marko Oesting, Kirstin Strokorb

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RandomFields

package for inference on and prediction and simulation of random fields.

Structure

- commands
  - `RFsimulate`, `RFfit`, `RFinterpolate`
  - ...
- models
  - 30 simple ones
    `RMexp`, `RMspheric`, `RMwhittle`, `RMgencauchy`, ...
  - 30 operators (functionals) used within a modular conception
    `+`, `RMmult`, `RMmatrix`, `RMdelay`, ...
- processes
  - `RPgauss` [default], `RPchi2`, `RPcirculant`, ...
- distributions (hierarchical modelling)
  - `RRdistr` [automatic detection]

```
RFgetModelNames(group.by=c('operator', 'domain'))
```

Current version

Please install version 3.0.26 from

http://ms.math.uni-mannheim.de/de/publications/software/

(download and install from local directory)

see also for a function to help installing the package

`RandomFields_install.txt`

(automatic installation)

Warm-up examples

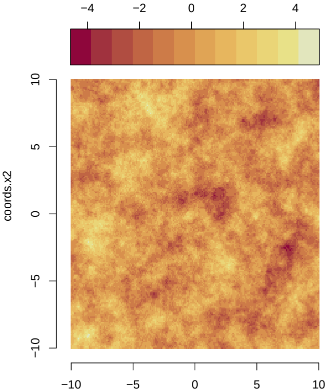
we use

```
x <- y <- seq(-10, 10, 0.1)
```

Warm-up examples [cont'd]

Basic exponential model $C(h) = e^{-\|h\|}$

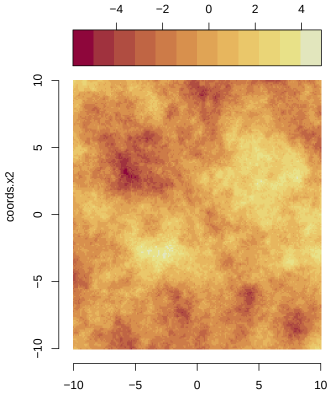
```
model <- RMexp()
simu <- RFsimulate(model, x, y, grid=TRUE)
plot(simu)
```



Warm-up examples [cont'd]

Exponential model with scale and variance $C(h) = 3e^{-\|h\|/5}$

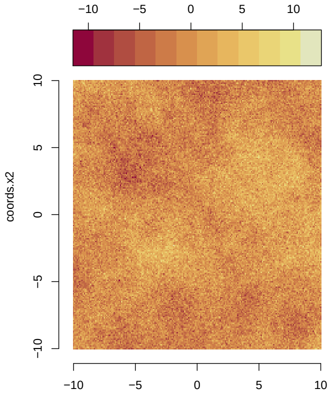
```
model <- RMexp(var=3, scale=5)
simu <- RFsimulate(model, x, y, grid=TRUE)
plot(simu)
```



Warm-up examples [cont'd]

Exponential model, also with nugget $C(h) = 3e^{-\|h\|/5} + 4 \cdot \mathbf{1}(h = 0)$

```
model <- RMexp(var=3, scale=5) + RMnugget(var=4)
simu <- RFsimulate(model, x, y, grid=TRUE)
plot(simu)
```



Example 1

Linear model of coregionalization Z

$$Z = MY = \sum_{j=1}^{k} M_j Y_j$$

$Y_1, \ldots, Y_k$ independent, univariate random fields, $Y = (Y_1, \ldots, Y_k)^\top$
$M = (M_1, \ldots, M_k)$ a $k \times k$ matrix; $M_1, \ldots, M_k \in \mathbb{R}^k$
Cross-covariance function equals $MCM^\top$
C : diagonal matrix containing the covariance functions $Y_1, \ldots, Y_k$.

(Whittle-)Matérn covariance function

$$W_\nu(h) = 2^{1-\nu}\Gamma(\nu)^{-1}\|h\|^\nu K_\nu(\|h\|), \qquad h \in \mathbb{R}^d.$$

K is a modified Bessel function
Γ the Gamma function,
$\nu > 0$ a smoothness parameter

Example 1 [cont'd]

Let $k = 2$,

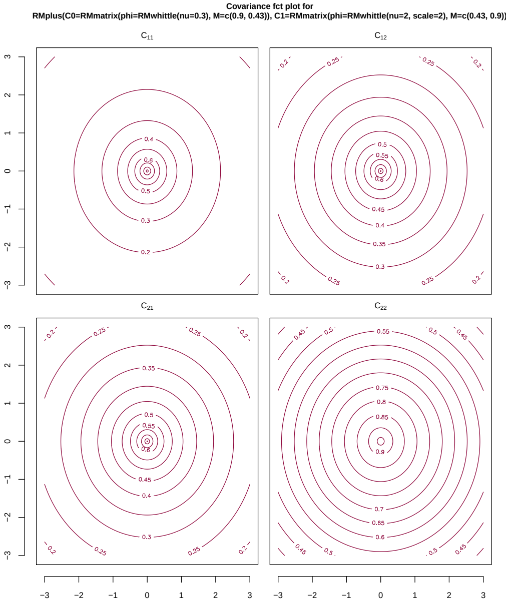
$$M_1 = \begin{pmatrix} 0.9 \\ 0.43 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0.43 \\ 0.9 \end{pmatrix},$$

Y_1, Y_2 Gaussian random fields with covariance function $W_{0.3}(h)$ and $W_2(h/2)$, respectively.

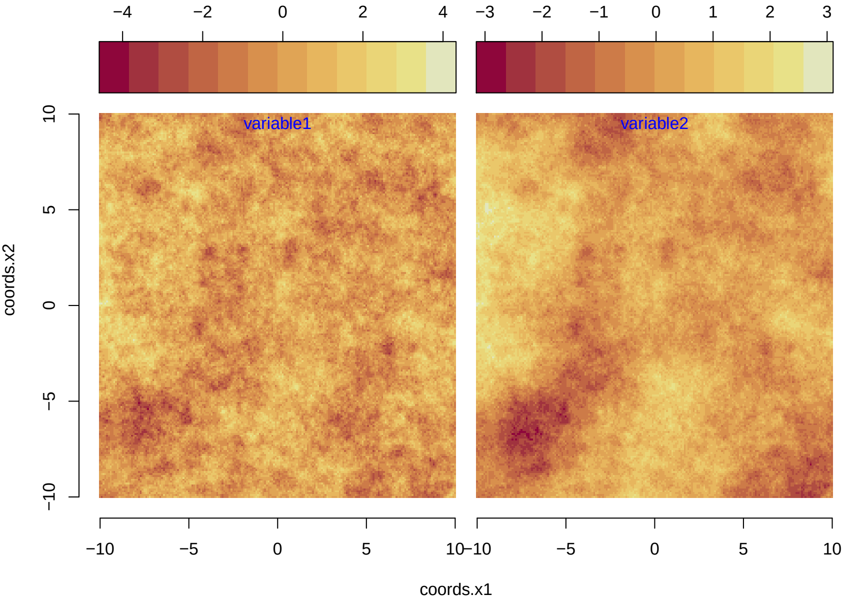
Code

```
R> M1 <- c(0.9, 0.43)
R> M2 <- c(0.43, 0.9)
R> model <- RMmatrix(M = M1, RMwhittle(nu = 0.3)) +
+           RMmatrix(M = M2, RMwhittle(nu = 2, scale = 2))
R> plot(model, dim = 2, xlim = c(-3, 3))
R> x <- y <- seq(-10, 10, 0.2)
R> simu <- RFsimulate(model, x, y, grid = TRUE)
R> plot(simu)
```

Example 1 [cont'd]



Example 1 [cont'd]



Example 2

Bivariate Whittle-Matérn class (?)

All 4 componentents of the cross covariance function

$$\begin{pmatrix} W_{11}(h) & W_{12}(h) \\ W_{12}(h) & W_{22}(h) \end{pmatrix}$$

are Whittle-Matérn functions,

$$W_{ij}(h) = c_{ij} \frac{2^{1-\nu_{ij}}}{\Gamma(\nu_{ij})} (\|h\|/s_{ij})^{\nu_{ij}} \mathrm{K}_{\nu_{ij}}(\|h\|/s_{ij}), \qquad i,j = 1,2,$$

where $c_{ii} \geq 0$, $c_{ij} = c_{ji}$, $s_{ij} = s_{ji} > 0$, $\nu_{ij} = \nu_{ji} > 0$,

Example 2

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where $c_{ii} \geq 0$, $c_{ij} = c_{ji}$, $s_{ij} = s_{ji} > 0$, $\nu_{ij} = \nu_{ji} > 0$,
$\nu_{12} = \nu_{\mathrm{red}} \frac{\nu_{11}+\nu_{22}}{2}$ for $\nu_{\mathrm{red}} \geq 1$
$c_{12} = \rho_{\mathrm{red}} c_0$ for some constant $c_0 > 0$ and $\rho_{\mathrm{red}} \in [-1,1]$

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$\nu_{12} = \nu_{\mathrm{red}} \frac{\nu_{11}+\nu_{22}}{2}$ for $\nu_{\mathrm{red}} \geq 1$
$c_{12} = \rho_{\mathrm{red}} c_0$ for some constant $c_0 > 0$ and $\rho_{\mathrm{red}} \in [-1,1]$

$$c_0^2 = c_{11} c_{22} \frac{\Gamma(\nu_{11}+\frac{d}{2})\Gamma(\nu_{22}+\frac{d}{2})\Gamma(\nu_{12})^2}{\Gamma(\nu_{11})\Gamma(\nu_{22})\Gamma(\nu_{12}+\frac{d}{2})^2} \frac{s_{11}^{d/2} s_{22}^{d/2}}{s_{12}^d} \inf_{t \geq 0} \frac{(1+s_{12}^2 t^2)^{2\nu_{12}+d}}{(1+s_1^2 t^2)^{\nu_{11}+d/2}(1+s_2^2 t^2)^{\nu_{22}+d/2}}$$

Example 2

Bivariate Whittle-Matérn class (?)

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are Whittle-Matérn functions,

$$W_{ij}(h) = c_{ij} \frac{2^{1-\nu_{ij}}}{\Gamma(\nu_{ij})} (\|h\|/s_{ij})^{\nu_{ij}} \mathrm{K}_{\nu_{ij}}(\|h\|/s_{ij}), \qquad i,j = 1,2,$$

where $c_{ii} \geq 0$, $c_{ij} = c_{ji}$, $s_{ij} = s_{ji} > 0$, $\nu_{ij} = \nu_{ji} > 0$,
$\nu_{12} = \nu_{\mathrm{red}} \frac{\nu_{11}+\nu_{22}}{2}$ for $\nu_{\mathrm{red}} \geq 1$
$c_{12} = \rho_{\mathrm{red}} c_0$ for some constant $c_0 > 0$ and $\rho_{\mathrm{red}} \in [-1, 1]$

Example 2 [cont'd]

$\nu_{11} = 1$, $s_{11} = 1$, $c_{11} = 1$
$\nu_{22} = 2$, $s_{22} = 2$, $c_{22} = 5$

Example 2 [cont'd]

$\nu_{11} = 1$, $s_{11} = 1$, $c_{11} = 1$
$\nu_{22} = 2$, $s_{22} = 2$, $c_{22} = 5$

$\nu_{\mathrm{red}} = 1$, i.e. $\nu_{12} = \nu_{\mathrm{red}}(\nu_{11} + \nu_{22})/2 = 1.5$
$s_{12} = 1$
$\rho_{\mathrm{red}} = 1$, i.e. $c_{12} = 0.527$ and $\frac{c_{12}}{\sqrt{c_{11} c_{22}}} = \frac{0.527}{\sqrt{1 * 5}} = 0.2357$

Example 2 [cont'd]

$\nu_{11} = 1$, $s_{11} = 1$, $c_{11} = 1$
$\nu_{22} = 2$, $s_{22} = 2$, $c_{22} = 5$

$\nu_{\mathrm{red}} = 1$, i.e. $\nu_{12} = \nu_{\mathrm{red}}(\nu_{11} + \nu_{22})/2 = 1.5$
$s_{12} = 1$
$\rho_{\mathrm{red}} = 1$, i.e. $c_{12} = 0.527$ and $\frac{c_{12}}{\sqrt{c_{11} c_{22}}} = \frac{0.527}{\sqrt{1 * 5}} = 0.2357$

Code

```
R> model <- RMbiwm(nudiag = c(1, 2), nured = 1, rhored = 1,
+                  cdiag = c(1, 5), s = c(1, 1, 2))
R> plot(model, dim = 2, xlim = c(-3, 3))
R> simu <- RFsimulate(model, x, y, grid = TRUE)
R> plot(simu)
```

Example 2 [cont'd]

$\nu_{11} = 1$, $s_{11} = 1$, $c_{11} = 1$
$\nu_{22} = 2$, $s_{22} = 2$, $c_{22} = 5$

$\nu_{\text{red}} = 1$, i.e. $\nu_{12} = \nu_{\text{red}}(\nu_{11} + \nu_{22})/2 = 1.5$
$s_{12} = 1$
$\rho_{\text{red}} = 1$, i.e. $c_{12} = 0.527$ and $\frac{c_{12}}{\sqrt{c_{11} c_{22}}} = \frac{0.527}{\sqrt{1*5}} = 0.2357$

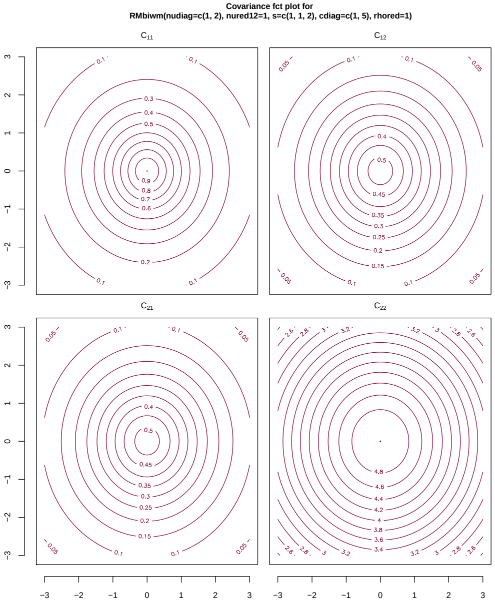
Code

```
R> model <- RMbiwm(nudiag = c(1, 2), nured = 1, rhored = 1,
+                  cdiag = c(1, 5), s = c(1, 1, 2))
R> plot(model, dim = 2, xlim = c(-3, 3))
R> simu <- RFsimulate(model, x, y, grid = TRUE)
R> plot(simu)
```

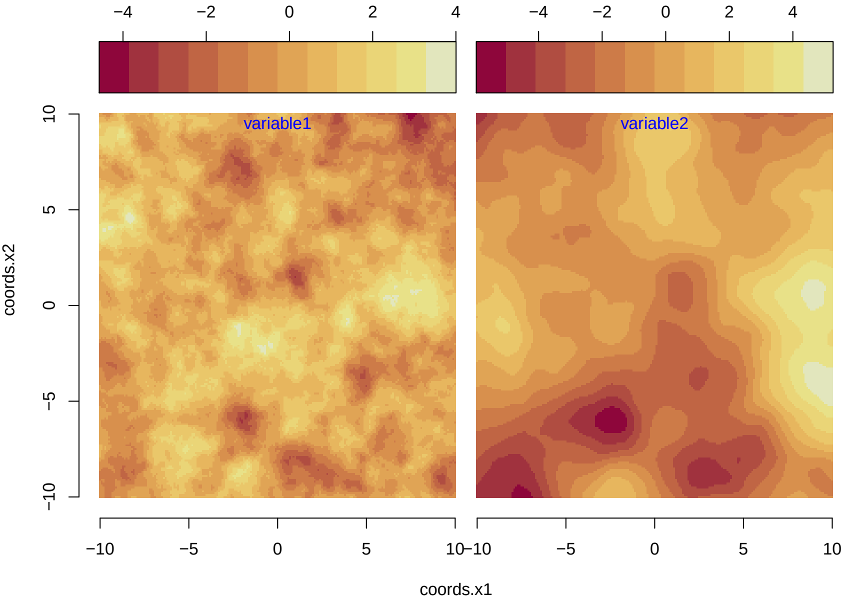
Value of c_{12}

```
R> RFgetModel()
```

Example 2 [cont'd]



Example 2 [cont'd]



Example 3: curl-free fields and divergence free vector fields

$F = (F_1, \ldots, F_d)$ be a vector field on $\mathbb{R}^d$.

$$\nabla \times F = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) e_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) e_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) e_3 \quad (d = 3)$$

$$\nabla \cdot F = \sum_{i=1}^{d} \frac{\partial F_i}{\partial x_i}.$$

Helmholtz decomposition for $d = 3$

Under mild assumption

$$F = F^c + F^d$$

with

$$\nabla \times F^c = 0 \qquad \text{(curl free field)}$$

and

$$\nabla \cdot F^d = 0 \qquad \text{(divergence free field)}$$

Example 3 [cont'd]

Random vector random fields

C_0: covariance function (of a stationary rf), sufficiently often differentiable
Then (nearly "iff")

$$C^d(h) = (-\Delta E + \nabla\nabla^T)C_0(h)$$

belongs to a (stationary) divergence free random field and

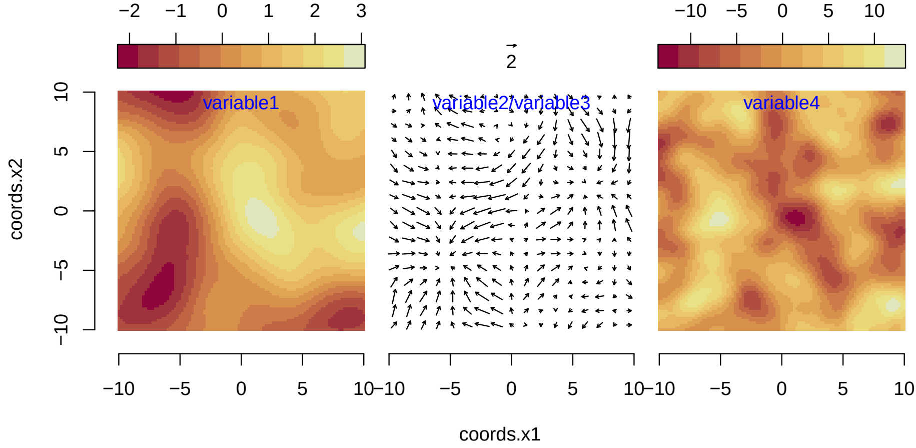
$$C^c(h) = (-\nabla\nabla^T)C_0(h)$$

belongs to a (stationary) rotation free random field.

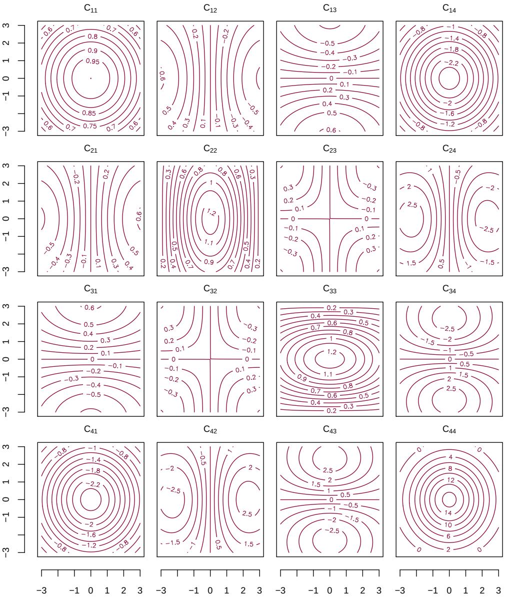
Code

```
R> model <- RMcurlfree(RMmatern(nu = 5), scale = 4)
R> plot(model, dim = 2, xlim = c(-3, 3))
R> simu <- RFsimulate(model, x, y, grid = TRUE)
R> plot(simu, select.variables = list(1, 2 : 3, 4))
```

Example 3 [cont'd]



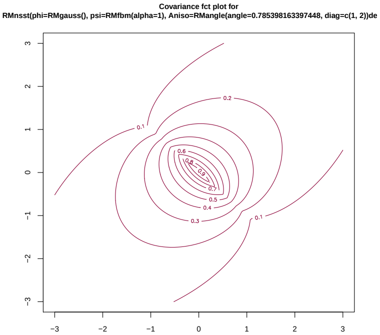
Example 3 [cont'd]



Example 4: nonstationary space-time model

Code for genuine space-time model, rotated

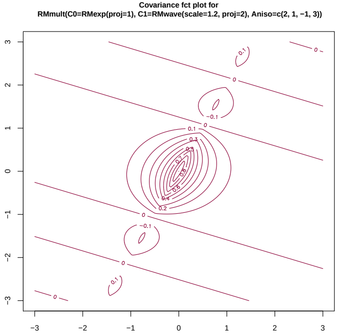
```
R> Aniso <- RMangle(pi / 4, diag = c(1, 2))
R> nsst <- RMnsst(Aniso=Aniso,
+                 phi=RMgauss(),
+                 psi=RMfbm(alpha=1), delta=2)
R> plot(nsst, xlim=c(-3,3), dim=2)
```



Example 4 [cont'd]

Code for separable model, rotated

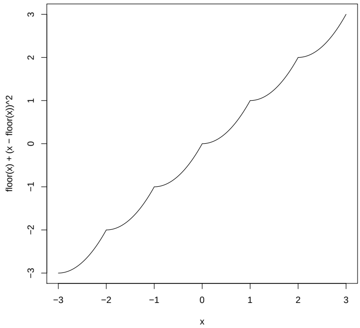
```
sep <- RMmult(RMexp(proj=1), RMwave(proj=2, scale=1.2),
              Aniso=matrix(ncol=2, c(2, 1, -1, 3)))
plot(sep, xlim=c(-3,3),  dim=2)
simu <- RFsimulate(sep, x,x)
plot(simu)
```

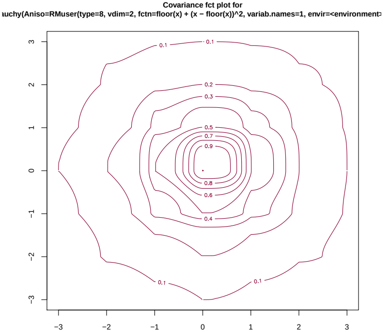


Example 4 [cont'd]

Code for space deformation

```
R> Aniso2 <- RMuser(fctn=floor(x) + (x - floor(x))^2, vdim=2)
R> nonstat <- RMcauchy(Aniso = Aniso2, gamma=1)
R> plot(nonstat, xlim=c(-3,3), dim=2)
```

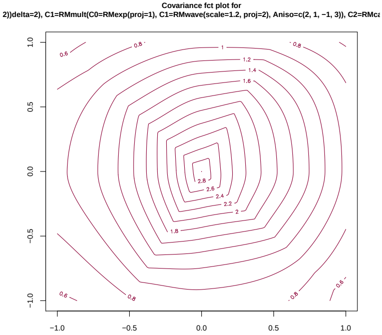




Example 4 [cont'd]

Code

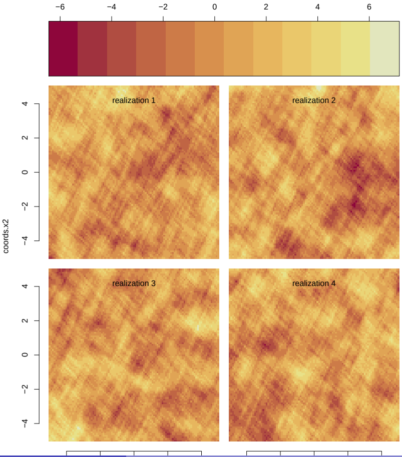
```
R> model <- nsst + sep + nonstat
R> plot(model, xlim=c(-1,1), dim=2)
```



Example 4 [cont'd]

Code

```
R> x <- y <- seq(-4, 4, 0.1)
R> simu <- RFsimulate(model, x=x, y=x, n=4)
R> plot(simu)
```



Example 5: Generalised Paciorek-Schervish model

Model

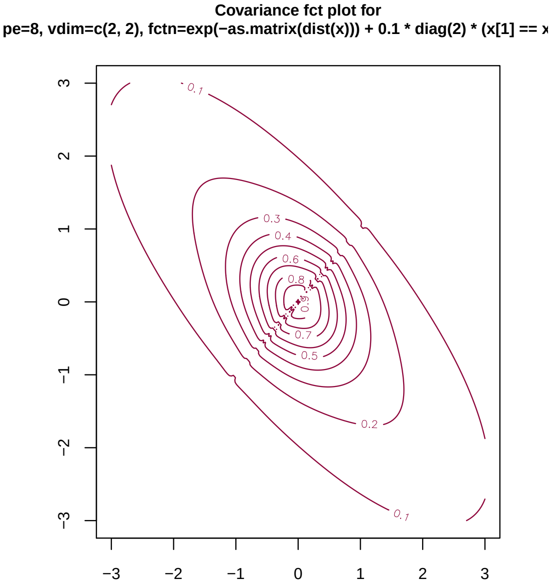
$$
\begin{aligned}
C(x,y) &= |S_x|^{1/4}|S_y|^{1/4}|A|^{-1/2}\varphi(Q(x,y)^{1/2}), \qquad \text{where} \\
Q(x,y) &= c^2 - m^2 + h^\top (S_x + 2(m+c)M)A^{-1}(S_y + 2(m-c)M)h \\
c &= -z^\top h + \xi(x) - \xi(y), \\
A &= S_x + S_y + 4Mhh^\top M \\
m &= h^\top M h \\
h &= x - y \\
\varphi &: \text{normal scale mixture} \\
S &: \mathbb{R}^d \to \mathbb{R}^{d\times d}, \text{positive definite matrices} \\
\xi &: \mathbb{R}^d \to \mathbb{R} \qquad [\text{ Paciorek-Schervish} : \xi \equiv 0 \text{ }] \\
z &\in \mathbb{R}^d \qquad [\text{ Paciorek-Schervish} : z = 0 \text{ }] \\
M &\in \mathbb{R}^{d\times d}, \text{symmetric} \qquad [\text{ Paciorek-Schervish} : M = 0 \text{ }]
\end{aligned}
$$

Example 5 [cont'd]

Code

```
R> S <- RMuser(fctn=exp(-as.matrix(dist(x)))
+                  + 0.1 * diag(2) * (x[1]==x[2]),
+               vdim=c(2, 2))
R> model <- RMstp(xi=RMgauss(),
+                  phi = RMstable(alpha=1.5),
+                  z = rep(0, 2),
+                  S = S,
+                  M = matrix(ncol=2, nrow=2, 0)
+                  )
R> plot(model, xlim=c(-3, 3), dim=2)
R> x <- y <- seq(-8, 8, 0.2)
R> simu <- RFsimulate(model, x, y, grid = TRUE)
R> plot(simu)
```

Example 5 [cont'd]



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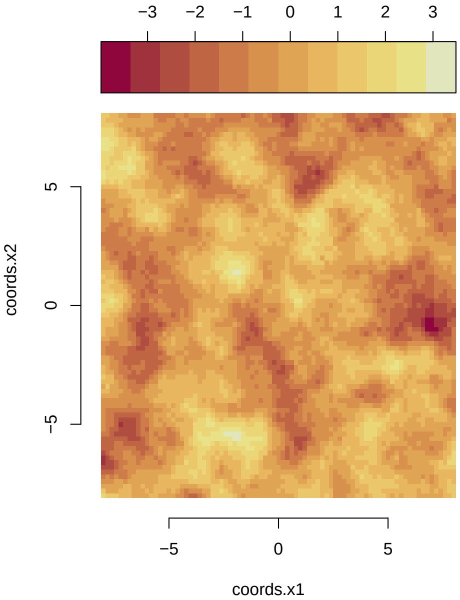
Example 5 [cont'd]



What can be done more with RandomFields?

- conditional simulation for Gaussian fields using `RFsimulate`
- paraneter estimation for Gaussian fields `RFfit`
- spatial prediction for Gaussian fields `RFinterpolate`
- likelihood ratio test for Gaussian fields
- cross validation for Gaussian fields
- simulation of fields related to the Gaussian fields, e.g. χ_2 fields
- simulation of max-stable random fields
- estimation of fractal dimension and Hurst effect