Non-gaussian Matérn fields

Applied to the housing data

David Bolin
Chalmers University of Technology

joint work with Jonas Wallin

Búzios, RJ, Brazil,
June 20, 2014

The SPDE approach

A Gaussian Matérn field is a solution to the SPDE

$$(\kappa^2 - \Delta)^{\frac{\alpha}{2}} X(\mathbf{s}) = \sigma \mathcal{W}(\mathbf{s})$$

where $\alpha = \nu + d/2$, $\mathcal{W}(\mathbf{s})$ is Gaussian white noise and Δ is the Laplacian[1]. Advantages with this representation:

- Defines Matérn fields on general smooth manifolds.
- Can be used to define non-stationary models by allowing the parameters in the SPDE to be spatially varying.
- Facilitates computationally efficient approximations through Hilbert space approximations.

[1] Lindgren, F., Rue, H., and Lindström, J.: An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach (with discussion); *JRSS Series B*, 2011

Non-stationary models

Introducing non-stationarity in the SPDE models is very easy[2]. We just allow the parameters of the usual model to be spatially varying:

$$(\kappa(s) - \Delta)x(s) = \sigma(s)W(s)$$

where

$$\log \sigma(s) = \sum_{i=1}^{p} B_i^{\sigma}(s)\theta_i, \qquad \log \kappa(s) = \sum_{i=1}^{p} B_i^{\kappa}(s)\theta_{i+p}.$$

The functions $B_i(s)$ are covariates or spatial basis functions.

The resulting covariance functions will automatically be valid.

[2] See Lindgren at al 2011, or Bolin and Lindgren: Spatial models generated by nested stochastic partial differential equations, with an application to global ozone mapping, *Annals of Applied Statistics*, 2011

Non-Gaussian Matérn fields

A solution to

$$(\kappa(s) - \Delta)x(s) = \sigma(s)W(s)$$

with $\log \sigma(s) = \sum_{i=1}^{p} B_i^{\tau}(s)\theta_i$ and priors for θ_i can either be viewed as a non-stationary Gaussian process or a non-Gaussian process.

What if we would take the basis expansion to the limit and let $\sigma(s)$ be a positive stochastic process?

A formal version of this idea is to replace $\sigma(s)W$ with a type-G Lévy process M:

$$(\kappa^2 - \Delta)^{\frac{\alpha}{2}} X = \dot{M}$$

Many of the nice properties of the SPDE method are preserved[3].

[3]Bolin, D.: Spatial Matérn fields driven by non-Gaussian noise; *Scandinavian Journal of Statistics*, 2013

Latent non-Gaussian models

- Two type-G processes have the properties we want
  - Generalized asymmetric Laplace (GAL) fields
  - Normal inverse Gaussian (NIG) fields
- We can use Matérn fields driven by noise processes of this type in a hierarchical model:[4]

$$Y_i = Z(s_i) + \mathcal{E}_i$$

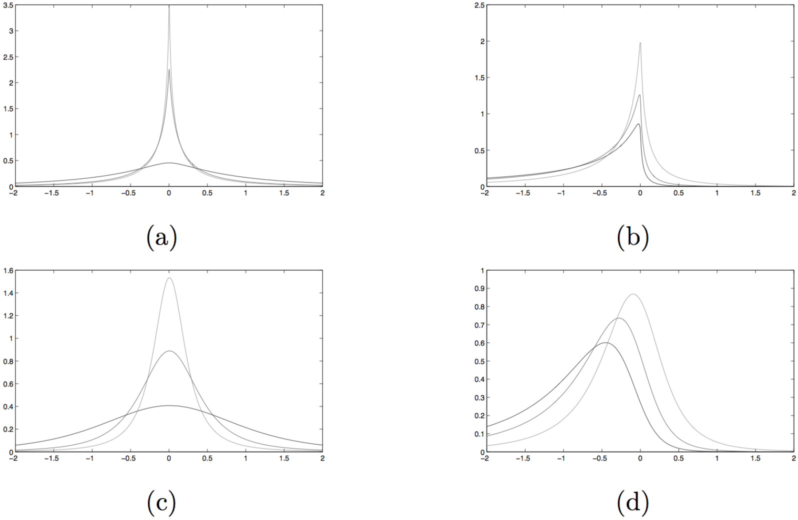
$$Z(s) = \sum_{i=1}^{p} b_i(s)\beta_i + X(s)$$

where

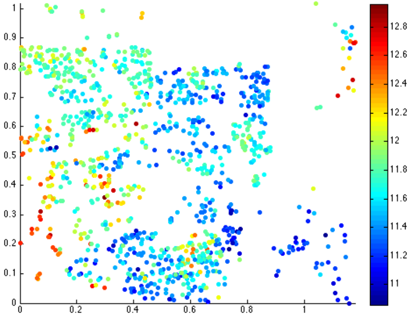
- Y_i is data
- $Z(s)$ is the latent field
- $\mathcal{E}_i$ is (Gaussian) measurement noise.
- $b_i(s)$ are covariates for the mean.
- $X(s)$ is the non-Gaussian Matérn field.

[4] Wallin, J. and Bolin, D.: Non-Gaussian Matérn fields with an application to precipitation modeling, ArXiv preprint, 2013

Examples of marginal distributions



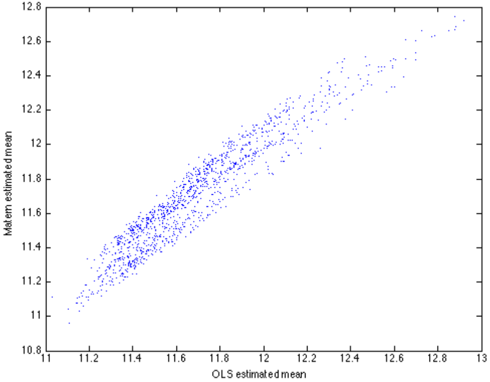
The housing data



- We will now compare a standard latent Gaussian Matérn model with latent non-Gaussian Matérn models.
- All models are stationary, though the non-Gaussian models (almost) could be viewed as a Gaussian model with (stochastic) spatially varying variances (and means).

First of all: Estimating the mean $\mu(s) = \sum_{i=1}^{p} b_i(s)\beta_i$

OLS estimate of mean shown against mean estimated jointly with a spatial Matérn model.



We should estimate the mean jointly with the covariance model, but let's subtract the OLS estimated mean from the data and assume zero mean for the rest of the analysis.

Results for the housing data

Before we look at covariances, lets do a simple residual analysis for the different models.

- Variance of kriging residuals:

Gaussian	GAL	NIG
0.0196	0.0134	0.0133

- We might be overfitting! Cross-validated residuals:

Gaussian	GAL	NIG
0.0202	0.0161	0.0161

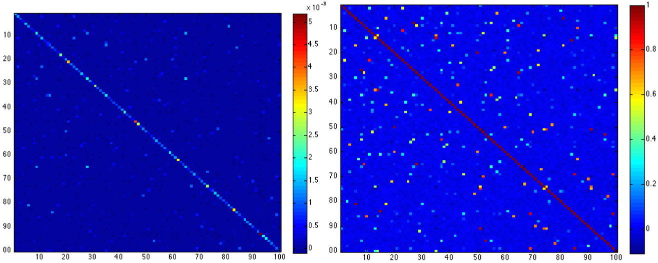
- results: NIG $\approx$ GAL and they are both better than Gaussian.

Estimated prior practical correlation ranges ($\sqrt{8\nu}\kappa^{-1}$):

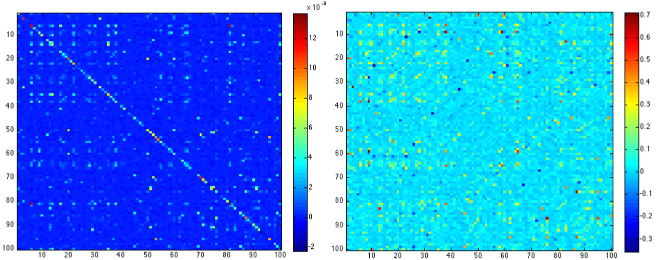
$$\text{Gaussian:} \quad 0.0256, \quad \text{NIG:} \quad 0.0259$$

Estimated posterior covariances and correlations

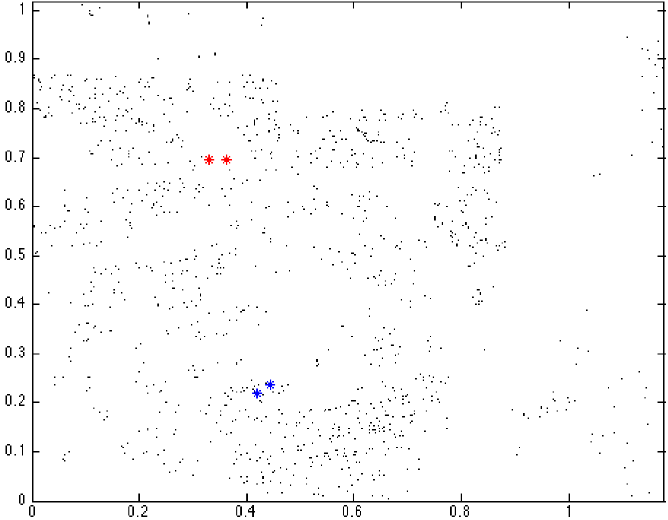
Estimates for the NIG model:



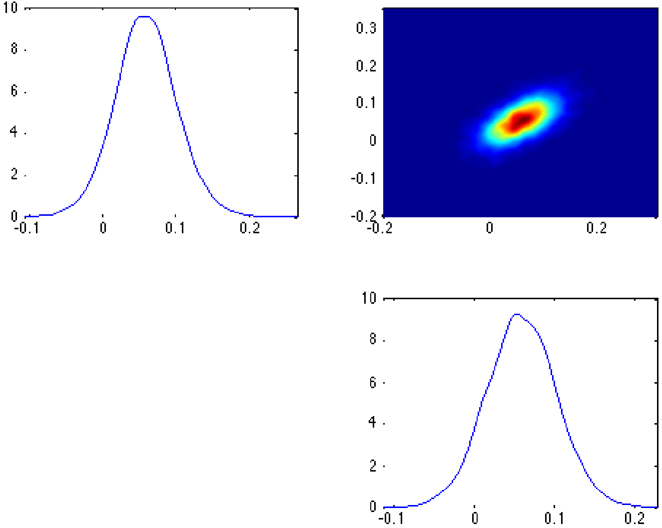
Differences with Gaussian Covariances and correlations:



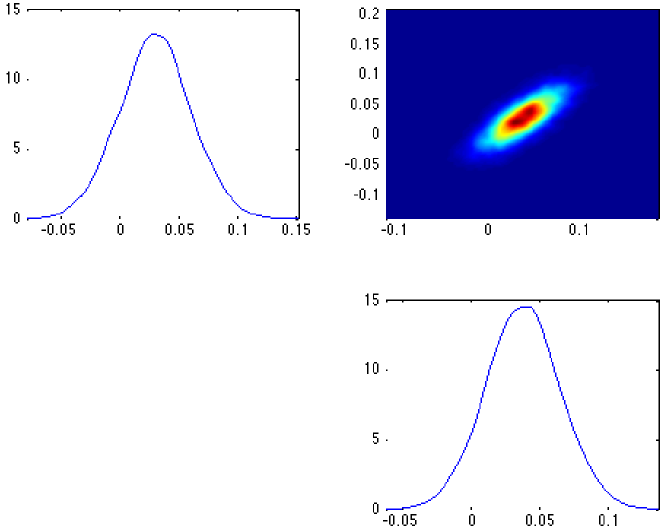
We also look at some predictive distributions



NIG predictions: First set of locations



NIG predictions: Second set of locations



Discussion

- Using non-Gaussian Matérn fields driven by NIG (or GAL) noise improves predictions.
  - Even more so if we estimate the mean jointly with the covariance structure.
  - And even more so if we model prices instead of log-prices.
- The reason is basically that the GAL process allows for spatially varying variances.
  - We only have two additional parameters in the non-Gaussian model (shape and asymmetry of the marginals).
- We could allow for non-stationarity in the parameters as well.
- Compared with non-stationary latent Gaussian models, latent non-Gaussian models are much less common in geostatistics.
- I believe that we sometimes use (complicated) non-stationary models to capture (simpler) non-Gaussian features.