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Bayes Factors for Non-homogeneous Poisson Processes with Vague Prior Information

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SUMMARY

The calculation of Bayes factors for rival parametric models of non-homogeneous Poisson processes is considered. If vague prior information for the model parameters is represented by limiting improper prior forms, then the resulting Bayes factor is defined only up to a multiplicative constant. It is noted that the procedure proposed by Spiegelhalter and Smith (1982) does not provide any solution to the problem of assigning the constant if the priors are inappropriately specified, and sufficient conditions for it to provide a solution are derived. It is shown that priors which satisfy the conditions exist in most situations. The case of monotonic log-polynomial intensity models is then considered in some detail.

Keywords: BAYES FACTORS; HAAR PRIOR; IMAGINARY OBSERVATIONS; LOG-POLYNOMIAL INTENSITY FUNCTION; NON-HOMOGENEOUS POISSON PROCESS; VAGUE PRIOR INFORMATION

1. INTRODUCTION

The purpose of this paper is to develop a Bayesian approach to the problem of comparing models for non-homogeneous Poisson processes. We assume that the Poisson process has a rate or intensity function, $\lambda(s)$, for which rival parametric models

$$M_i: \lambda(s) = \lambda_i(s, \theta_i) \quad (i = 0, 1) \tag{1.1}$$

have been postulated, where s denotes time. The comparison is to be based on an observation period $[0, T]$ during which n events have occurred at times $t = (t_1, \ldots, t_n)$. Here we shall be concerned with the case where the priors have improper limiting forms representing vague prior information for each of the models.

Bayesian comparison of models under vague prior information has been discussed by Spiegelhalter and Smith (1982)—hereafter SS—who review previous literature, San Martini and Spezzaferri (1984), and Perrichi (1984). Much of this literature has focussed on linear models, but, as we shall see, the Poisson process case, which does not seem to have been considered before in this context, has features which are not present in the linear model case.

In what follows, we shall denote the (possibly vector) parameter of M_i by θ_i, the parameter space by Θ_i, the prior measure by Π_i, and the prior density, if it exists, by p_i. The Bayes factor for M_0 against M_1 is

$$B_{01}^{(n)}(t, T) = p(t \mid M_0)/p(t \mid M_1) \tag{1.2}$$

the ratio of the marginal likelihoods, which we shall denote by B_{01} if this is unambiguous. Here

$$p(t \mid M_i) = \int p(t \mid \theta_i, M_i)\Pi_i(d\theta_i). \tag{1.3}$$

When the priors are improper, B_{01} is defined only up to an arbitrary ratio of constants.

To fix the ratio of constants we shall adopt the proposal of SS and carry out a "thought experiment" leading to an imaginary training sample. The basic idea is to imagine that a data set is available which involves the smallest possible sample size permitting a comparison of M_0 and M_1 and provides maximum possible support for M_0, and then to argue that the resulting

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B_{01} should be only slightly greater than one. It is important that the priors be appropriately chosen. For example, SS have pointed out that for linear models, the choice, made by several previous authors, of a vague prior limit of independent priors for the parameters may lead to a lack of invariance to scale changes in the dependent variable.

In the Poisson process case, such a choice of priors may have even more serious consequences, as it can lead to a situation in which the SS proposal does not provide any solution to the problem of assigning the ratio of constants. For example, consider the family of monotonic log-polynomial intensity function models

$$M_i : \lambda(s) = \lambda_i(s, \theta_i) = \theta_{i0} \exp\left(- \sum_{k=1}^{i} \theta_{ik} s^k \right) \quad (i = 0, 1, \ldots) \tag{1.4}$$

where $\theta_i = (\theta_{i0}, \theta_{i1}, \ldots, \theta_{ii})$ and $\theta_{ik} \geqslant 0$ for each i, k. Suppose we wish to compare M_0 with M_1, i.e. a constant with a log-linear intensity function model. Suppose $p_0(\theta_{00}) = c_0 \theta_{00}^{-1}$, as suggested by Lindley (1965), and $p_1(\theta_{10}, \theta_{11}) = c_1 \theta_{10}^{-1}$, obtained as a diffuse limit of conjugate priors. To implement the SS proposal, we now imagine a data set consisting of an observation period $[0, \tau]$ during which m events occur, where m is the smallest number of events needed to compare M_0 with M_1. It can be shown that, if the data provide maximal support for M_0, then the resulting value of B_{01} is proportional to τ. Thus, since τ is arbitrary, the SS proposal fails to assign a value to the ratio of constants. When the models and priors are such that the SS proposal does assign a value to the ratio of constants, we say that the resulting Bayes factor is *operational.*

In Section 2 we show that, under fairly general conditions, for non-homogeneous Poisson processes, a sufficient condition for the Bayes factor to be operational is that it be time-invariant, i.e. invariant to scale changes in the time variable. We then show that priors which yield time-invariant Bayes factors can be found in most situations. In section 3 we consider the comparison of monotonic log-polynomial intensity function models. In Section 4 we consider the log-linear intensity model in more detail, and in Section 5 we give two examples.

2. OPERATIONAL BAYES FACTORS FOR POISSON PROCESS MODELS

Let $U_k(\tau) = \{u = (u_1, \ldots, u_k): 0 \leqslant u_1 \leqslant u_2 \leqslant \ldots \leqslant u_k \leqslant \tau\}$. We say that the Bayes factor $B_{01}^{(n)}(t, T)$ is *operational* if there is a positive integer k such that

$$\sup_{\tau > 0} \sup_{u \in U_k(\tau)} B_{01}^{(k)}(u, \tau) < \infty.$$

Then, m, the smallest such integer, is the smallest number of events needed for a comparison of M_0 and M_1. We say that the Bayes factor B_{01} is *time-invariant* if, for $a > 0$, $B_{01}^{(n)}(at, aT) = B_{01}^{(n)}(t, T)$ for all n, t, T.

Theorem 1. Suppose that for each n and τ fixed, $B_{01}^{(n)}(u, \tau)$ is a bounded function of u. Suppose also that B_{01} is time-invariant. Then B_{01} is operational.

Proof.
$$\sup_{u \in U_k(\tau_1)} B_{01}^{(k)}(u, \tau_1)$$
$$= \sup_{u \in U_k(\tau_1)} B_{01}^{(k)}(au, a\tau_1), \text{ by the time-invariance of } B_{01}$$
$$= \sup_{v \in U_k(\tau_2)} B_{01}^{(k)}(v, \tau_2), \text{ where } \tau_2 = a\tau_1.$$

Thus $\sup_{u \in U_k(\tau)} B_{01}^{(k)}(\mu, \tau)$ is constant as a function of τ, and since it is finite by the boundedness assumption, the Bayes factor is operational. This completes the proof of Theorem 1. Note that for the boundedness assumption in Theorem 1 to hold, we must have $\lambda_1(s, \theta_1) > 0$ for all $s > 0$.

We now show that priors which yield time-invariant Bayes factors do exist, under fairly general conditions. Let Θ_i be a locally compact topological space. We define a group operation o_i which acts on the left of Θ_i, so that $G_i = (\Theta_i, o_i)$ is a locally compact topological group. Then, if B_i denotes the σ-algebra of Borel sets of G_i, there is a left Haar measure Π_i on (G_i, B_i) which is unique up to a multiplicative constant; see, for example, Halmos (1950, p. 263).

Theorem 2. Suppose that, for $i = 0, 1$, Π_i is the left Haar measure on (G_i, B_i), that the posterior resulting from the use of Π_i as a prior is always integrable, and that for each $a > 0$ there is a $g_i \in G_i$ such that $a\lambda_i(as, \theta_i) = \lambda_i(s, g_i o_i \theta_i)$, $(s \geqslant 0)$. Then B_{01} is time-invariant.

Proof. $B_{01}^{(n)}(at, aT)$ is the ratio of the integrals

$$\int_{G_i} \left\{ \prod_{j=1}^{n} \lambda_i(at_j, \theta_i \right\} \exp\left\{ - \int_0^{aT} \lambda_i(v, \theta_i) dv \right\} \Pi_i(d\theta_i)$$

$(i = 0, 1)$. Multiplying by a^n, letting $w = v/a$ and using the definition of g_i shows that this is the ratio of the integrals

$$\int_{G_i} \left\{ \prod_{j=1}^{n} \lambda_i(t_j, g_i o_i \theta_i) \right\} \exp\left\{ - \int_0^{T} \lambda_i(w, g_i o_i \theta_i) dw \right\} \Pi_i(d\theta_i)$$

$(i = 0, 1)$. If we replace θ_i by $g_i^{-1} o_i \theta_i$ and use the fact that Π_i is Haar, this becomes the ratio of the integrals

$$\int_{g_i^{-1} o_i G_i} \left\{ \prod_{j=1}^{n} \lambda_i(t_j, \theta_i) \right\} \exp\left\{ - \int_0^{T} \lambda_i(w, \theta_i) dw \right\} \Pi_i(d\theta_i) \tag{2.1}$$

$(i = 0, 1)$. Since G_i is a group, $g_i^{-1} o_i G_i = G_i$, so that the ratio of the integrals in (2.1) is $B_{01}^{(n)}(t, T)$. This completes the proof of Theorem 2.

3. BAYES FACTORS FOR LOG-POLYNOMIAL INTENSITY MODELS

We now consider the comparison of monotonic log-polynomial models of different degrees for the intensity function, as defined by (1.4). Non-Bayesian approaches to the problem, based on hypothesis testing, have been considered by Cox and Lewis (1966), Lewis (1972), MacLean (1974), Berman (1981), and Mathers (1984).

We shall denote the Bayes factor for M_i against M_j by $B_{ij}^{(n)}(t, T)$, or B_{ij} if this is unambiguous. We restrict attention to B_{0i}, without any loss of generality since $B_{ij} = B_{0j}/B_{oi}$. This relation holds if the priors are proper, but the SS procedure does not necessarily produce Bayes factors which obey it exactly, since different minimal experiments may be necessary for creating each pairwise comparison. Here we use it to mean that if the values of B_{0i} and B_{0j} yielded by the SS procedure are good approximations to the Bayes factors which would have been obtained using the proper, non-informative, priors anticipated from careful assessment, then their quotient would also be a good approximation to B_{ij} in the same sense.

We assume that $p_0(\theta_{00}) = c_0 \theta_{00}^{-1}$, and that $p_i(\theta_i)$ has the conjugate form

$$p_i(\theta_i) = c_i \theta_{i0}^{b_{i0}} \exp\left\{ - \sum_{k=1}^{i} b_{ik} \theta_{ik} - b_{i,i+1} \mu_i(\theta) \right\}, \tag{3.1}$$

where $\mu_i(\theta_i) = \int_0^T \lambda_i(s, \theta_i) ds$, the expected number of events given θ_i.

Theorem 3. B_{0i} is time-invariant if and only if $b_{i0} = -\{\frac{1}{2}i(i+1)+1\}$ and $b_{ik} = 0 (k = 1, \ldots, i)$ in (3.1).

Proof. The likelihood for M_i is

$$p(t \mid \theta_i, M_i) = \left\{\prod_{j=1}^{n} \lambda_i(t_j, \theta_i)\right\} \exp(-\mu_i(\theta_i)).$$

Thus, by (1.3)

$$p(t \mid M_0) = c_0(n - 1)!T^{-n}$$

$$p(t \mid M_i) = c_i \int \theta_{i0}^{n+b_{i0}} \exp\left\{-\sum_{k=1}^{i} \theta_{ik} S_{ik} - (b_{i,i+1} + 1)\mu_i(\theta_i)\right\} d\theta_{i0} d\theta_{i1} \ldots d\theta_{ii},$$

where $S_{ik} = \sum_{j=1}^{n} t_j^k + b_{ik}$, the integration with respect to each of the θ_{ik}'s being over the range $[0, \infty)$. Then, integration with respect to θ_{i0} yields

$$p(t \mid M_i) = c_i \Gamma(n_i)\{(b_{i,i+1} + 1)T\}^{-n_i} \int \exp\left\{-\sum_{k=1}^{i} \theta_{ik} S_{ik}\right\} h_i(\theta_i, T)^{n_i} d\theta_{i1} \ldots d\theta_{ii} \tag{3.2}$$

where $n_i = n + b_{i0} + 1$ and

$$h_i(y, T) = \left\{\int_0^1 \exp\left(-\sum_{k=1}^{i} y_k T^k v^k\right) dv\right\}^{-1},$$

$y = (y_1, \ldots, y_i)$ being an i-vector. Now suppose $a > 0$. Then, by (3.2)

$$B_{0i}^{(m)}(at, aT) = (c_0/c_i)(n - 1)! \; \{(b_{i,i+1} + 1)T\}^{n_i} \, \Gamma(n_i)^{-1} T^{-n} a^{b'_{i0}} I_{i1}^{-1}, \tag{3.3}$$

where $b'_{i0} = b_{i0} + \frac{1}{2}i(i + 1) + 1$,

$$I_{i1} = \int \exp\left\{-\sum_{k=1}^{i} \theta_{ik}(S_k + \rho_{ik})\right\} h_i(\theta_i, T)^{n_i} d\theta_{i1} \ldots d\theta_{ii},$$

$$S_k = \sum_{j=1}^{n} t_j^k, \text{ and } \rho_{ik} = b_{ik} a^{-k}.$$

Thus $B_{01}^{(n)}(at, aT) = B_{01}^{(n)}(t, T)$ $(a > 0)$ if and only if $b'_{i0} = 0$ and $\rho_{ik} = b_{ik} (k = 1, \ldots, i)$, i.e. if $b_{i0} = -\{\frac{1}{2}i(i + 1) + 1\}$ and $b_{ik} = 0$ $(k = 1, \ldots, i)$. This completes the proof of Theorem 3.

Thus, for each i there is a family of priors, indexed by $b_{i,i+1}$, each of which leads to a time-invariant Bayes factor. However, when $b_{i,i+1} \neq 0$ the prior depends on T, which may often be inappropriate. We thus set $b_{i,i+1} = 0$, leading to the priors $p_i(\theta_i) = c_i \theta_{i0}^{b_i}$, where $b_i = -\{\frac{1}{2}i(i + 1) + 1\}$ $(i = 0, 1, \ldots)$.

We now introduce the imaginary data $(u_1, \ldots, u_m)$, observed over the period $[0, \tau]$, where m is the smallest number of events that must be observed in order to permit a comparison between M_0 and M_i. By (3.2) the Bayes factor is

$$B_{0i}^{(m)}(u, \tau) = (c_0/c_i)(m - 1)! \; \tau^{(b_i + 1)} \Gamma(m')^{-1} I_{i2}^{-1},$$

where $I_{i2} = \int \exp\left(-\sum_{k=1}^{i} \theta_{ik} S'_k\right) h_i(\theta_i, \tau)^{m'} d\theta_{i1} \ldots d\theta_{ii}$, $m' = m + b_i + 1$, and $S'_k = \sum_{j=1}^{m} u_j^k$.

$B_{0i}^{(m)}(u, \tau)$ is maximised with respect to u when $S_k' = m\tau^k$ $(k = 1, \ldots, i)$, corresponding to the occurrence of all the m events at the end of the interval $[0, \tau]$. Thus

$$\sup_{u \in U_m(\tau)} B_{0i}^{(m)}(u, \tau) = (c_0/c_i)(m-1)\Gamma(m')^{-1} I_{i3}^{-1} \tag{3.4}$$

where $I_{i3} = \int \exp\left(-\sum_{k=1}^{i} m\theta_{ik}\right) h_i(\theta_i, 1)^{m'} d\theta_{i1} \ldots d\theta_{ii}$. m is then the smallest value of i for which (3.4) is defined, namely $m = \frac{1}{2}i(i+1)+1$. Setting (3.4) equal to one yields

$$(c_0/c_i) = \{(m-1)!\}^{-1} \int \exp\left(-\sum_{k=1}^{i} m\theta_{ik}\right) h_i(\theta_i, 1)\, d\theta_{i1} \ldots d\theta_{ii}. \tag{3.5}$$

By (3.2) the required Bayes factor is

$$B_{0i}^{(n)}(t, T) = (c_o/c_i)\{(n-1)\ldots(n-m+1)\} T^{(-m+1)} I_{i4}^{-1} \tag{3.6}$$

where (c_0/c_i) is defined by (3.5) and

$$I_{i4}^{-1} = \int \exp\left(-\sum_{k=1}^{i} \theta_{ik} S_k\right) h_i(\theta_i, T)^{(n-m+1)} d\theta_{i1} \ldots d\theta_{ii}.$$

4. COMPARING THE CONSTANT AND LOG-LINEAR INTENSITY MODELS

The simplest of the models considered in section 3 is the log-linear model M_1. This has been quite widely used, both because of its simplicity and the fact that it provides a physically motivated model for reliability growth; see Cozzolino (1968). Also, Langberg and Singpurwalla (1985) have pointed out that, under certain assumptions, M_1 is equivalent to the widely-used model for software reliability growth of Jelinski and Moranda (1972).

For this model, $m = 2$, so that, by (3.5)

$$c_0/c_1 = \int_0^\infty y e^{-2y}(1-e^{-y})^{-1} dy = \zeta(2) - 1 = 0.6449$$

where ζ denotes the Riemann zeta function. Thus, by (3.6),

$$B_{01}^{(n)}(t, T) = 0.6449(n-1)\left[\int_0^\infty e^{-Ry}\{y/(1-e^{-y})\}^{n-1} dy\right]^{-1} \tag{4.1}$$

where $R = S_1/T$.

M_1 can represent decreasing, but not increasing, rates of occurrence. We now indicate how our results generalise to the corresponding model, M_I, for an increasing intensity function, defined by (1.4) with $i = 1$ and $\theta_{11} < 0$. The appropriate minimal experiment is defined by $m = 2$ and $S_1' = 0$, so that

$$c_0/c_I = \int_0^\infty y(e^y - 1)^{-1} dy = \zeta(2) = 1.6449$$

and so

$$B_{0I}^{(n)}(t, T) = 1.6449(n-1)\left[\int_0^\infty e^{Ry}\{y/(e^y - 1)\}^{n-1} dy\right]^{-1} \tag{4.2}$$

(4.1) and (4.2) together yield a Bayes factor for M_0 against the model $M_C = M_1 \cup M_I$, where M_1 and M_I are given equal prior weight. M_C can represent changing rates of occurrence, both increasing and decreasing. The Bayes factor is as follows:

$$B_{0C} = 2B_{01}B_{0I}\{B_{01} + B_{0I}\}^{-1} \tag{4.3}$$

The standard tests of the null hypothesis M_0 against the alternative hypotheses M_1, M_I, and M_C are based on the statistic

$$U = \left(R - \frac{n}{2}\right)\left(\frac{n}{12}\right)^{-1/2} \tag{4.4}$$

This tends rapidly to a standard normal random variable; see Cox and Lewis (1966, p. 47). Jeffreys (1961) has suggested, for nested models $M_0 \subset M_1$ the following rough guidelines for the interpretation of B_{01}: if $B_{01} > 1$ the evidence supports M_0; if $B_{01} < 10^{-1}$ there is strong evidence against M_0; and if $B_{01} < 10^{-2}$ there is decisive evidence against M_0.

Table 1 gives values of the standard U-statistic (4.4) corresponding to critical values of B_{01} for $n = 10,20,50,100,200$. Table 2 gives values of U corresponding to critical values of B_{0I}. Table 3 gives upper and lower values of U corresponding to critical values of B_{0C}.

For all the sample sizes considered, the standard test suggests stronger evidence against M_0 than is implied by the Bayes factor. At sample sizes of 50 and above, evidence at a very high significance level is required for the Bayes factor to favour the more complex model strongly. This phenomenon is related to the Lindley paradox; see Lindley (1957).

A comparison with the results of SS shows that in the present, Poisson process, case, the discrepancy between the results of the standard test and the Bayes factor is even greater than in the linear model case. In general, the Bayes factors provide an automatic assessment of "significance", taking into account the amount of data available.

B_{0C} is not an even function of U, although this asymmetric effect rapidly becomes small as sample size increases. Also, the entries in Tables 1 and 2 are not equal in absolute value, so that inferences drawn are not completely invariant to time reversal of the process. While at first

TABLE 1
U equivalents of B_{01}

		B_{01}	
n	1	10^{-1}	10^{-2}
10	−1.69	−2.72	−3.35
20	−2.21	−3.08	−3.71
50	−2.73	−3.48	−4.08
100	−3.07	−3.75	−4.32
200	−3.37	−4.00	−4.54

TABLE 2
U equivalents of B_{0I}

		B_{0I}	
n	1	10^{-1}	10^{-2}
10	1.09	1.91	2.46
20	1.83	2.58	3.15
50	2.57	3.25	3.80
100	3.02	3.65	4.18
200	3.40	3.99	4.50

TABLE 3
Upper and lower U equivalents of B_{0C}

n	B_{0C} 1	10^{-1}	10^{-2}
10	(−2.06, 1.36)	(−2.93, 2.09)	(−3.50, 2.59)
20	(−2.50, 2.08)	(−3.28, 2.76)	(−3.87, 3.29)
50	(−2.98, 2.79)	(−3.67, 3.43)	(−4.24, 4.17)
100	(−3.29, 3.22)	(−3.93, 3.82)	(−4.48, 4.33)
200	(−3.58, 3.58)	(−4.17, 4.15)	(−4.69, 4.64)

sight surprising, this should not be too unexpected, since invariance of inferences to time reversal is a rather artificial requirement, corresponding to invariance to a non-linear transformation of the parameters which depends on the stopping time T.

5. EXAMPLES

Example 1. Coal-mining disasters

We now apply the results of the preceding sections to the data set consisting of times of occurrence of coal-mining disasters given by Jarrett (1979). Jarrett considered constant, log-linear, and log-quadratic intensity models for these data, and we now compare them. From (4.1), evaluating the integral numerically using the *IMSL* routine *DCADRE*, we found that B_{01} is 2.56×10^{-11}, which constitutes decisive evidence for the log-linear model against the simpler, constant intensity, model.

To find B_{02} we used (3.5) and (3.6) with $i = 2$ and $m = 4$. The double integrals were evaluated numerically using the *IMSL* routine *DBLIN*. This yielded $(c_0/c_2) = 0.0127$ and $B_{02} = 9.49 \times 10^{-8}$. Thus $B_{12} = 3.72 \times 10^3$, indicating that the data also provide strong support for the log-linear model over the more complicated, log-quadratic, model. These conclusions are similar to those of Jarrett (1979).

Example 2. Failures of airconditioning equipment

Proschan (1963) gave data, later reanalysed by Cox and Lewis (1966), on the times of failure of the airconditioning systems of each member of a fleet of thirteen Boeing 720 aircraft. One concern of Proschan (1963) was to ascertain whether the data provided evidence of decreasing failure rates. In Table 4, we address this concern by showing values of $\log_{10} B_{01}$ for each of the

TABLE 4
Bayes factors and values of U for the airconditioner data

Aircraft	*Serial no*	*n*	$\log_{10} B_{01}$	$\log_{10} B_{0I}$	$\log_{10} B_{0C}$	*U*
1	7907	5	0.51	−0.69	−0.41	0.04
2	7908	23	1.80	−0.77	−0.47	2.24
3	7909	29	1.00	1.98	1.26	−1.48
4	7910	15	0.95	1.23	1.06	−0.75
5	7911	14	1.30	0.08	0.36	0.97
6	7912	30	1.96	−0.39	−0.08	2.21
7	7913	27	1.34	1.65	1.47	−0.55
8	7914	24	1.49	1.18	1.31	0.26
9	7915	9	0.82	0.41	0.57	−0.14
10	7916	6	0.73	−1.09	−0.79	0.69
12	8044	12	1.24	−0.35	−0.06	1.17
13	8045	16	1.25	0.78	0.96	0.27

aircraft. Like Cox and Lewis (1966) we omit the eleventh aircraft (serial no. 7917), which experienced only two failures, and we ignore the fact that four of the aircraft underwent major overhauls, although a visual examination of the data suggests that the overhaul may have changed the failure pattern for the second aircraft (serial no. 7908). The values of B_{01} indicate that the data provide no support for the decreasing failure rate hypothesis.

In order to see whether there is any evidence that the failure rates were changing, we show values of B_{0I} and B_{0C}, together with values of U, as given in Cox and Lewis (1966, Table 3.4), for purposes of comparison. These were calculated as in (4.4), with n replaced by $(n - 1)$, and R replaced by $(R - 1)$, to take account of the fact that the period of observation ended with a failure. There is a discrepancy between their value of U and ours for the seventh aircraft (serial no. 7913), apparently due to the fact that they included the last failure time in their calculations for this aircraft.

The results show that there is little evidence of a changing failure rate for any of the aircraft. In particular, for the two aircraft for which the two-sided test based on U rejected the constant rate hypothesis at significance levels under 0.03 (the second, serial no. 7908, and the sixth, serial no. 7912), the Bayes factor provided only slight evidence for a changing rate. It would be possible to carry out a pooled analysis of the data for all the aircraft using the basic method proposed here. However, for this data set, such an exercise would not seem likely to add much to the analysis.

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