Calculus Review Guide

Stat 509 / Econ 580

This guide contains some basic results that you should have encountered during a basic course on calculus. This list is certainly not comprehensive: it is designed more to serve as an aid to memory.

1 Differentiation

1.1 Basics

$$\frac{d}{dx}e^{kx} = ke^{kx}$$
$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}x^k = kx^{k-1}$$

1.2 Chain Rule

Suppose that f and g are differentiable functions, then

$$h(t) = g(f(t))$$

is a differentiable function of t with

$$\left.\frac{dh}{dt}\right|_t = \left.\frac{dg}{df}\right|_{f(t)} \left.\frac{df}{dt}\right|_t$$

1.3 Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

1.4 Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

1.5 Stationary Points: function of one variable

The point x_0 is a *critical point* of the function $f(x)$ if

$$\frac{d}{dx}f(x) = 0.$$

- If $f''(x) < 0$ for all points x in a small interval $[x_0 - \epsilon, x_0 + \epsilon]$ around x_0, then x_0 is a local maximum of the function $f(x)$.
- If $f''(x) > 0$ for all x in a small interval $[x_0 - \epsilon, x_0 + \epsilon]$ then x_0 is a local minimum of the function $f(x)$.

1.6 Stationary points: function of two variables

Suppose that $g(\cdot, \cdot)$ is a function of the two variables x, y that has continuous derivatives $g_x(x, y) \equiv \frac{\partial}{\partial x} g(x, y)$ and $g_y(x, y) \equiv \frac{\partial}{\partial y} g(x, y)$ over some open and connected subset A of (x, y) values. Let (x_0, y_0) denote a point in A at which the partial derivatives vanish; that is,

$$0 = g_x(x_0, y_0) \quad \text{and} \quad 0 = g_y(x_0, y_0).$$

Such points are called *critical points*. Let D denote the matrix of second partial derivatives; thus

$$D \equiv \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial y\, \partial x} \\ \frac{\partial^2 g}{\partial x\, \partial y} & \frac{\partial^2 g}{\partial y^2} \end{pmatrix}.$$

Let $|D| = g_{xx}g_{yy} - g_{xy}g_{yx}$ be the determinant of D. Then g has a relative (or local) maximum at the critical point (x_0, y_0) provided

$$|D| > 0 \qquad \text{and both} \qquad g_{xx} < 0 \text{ and } g_{yy} < 0 \qquad \text{(all at } (x_0, y_0)), \tag{1}$$

while g has a relative minimum at the critical point (x_0, y_0) provided

$$|D| > 0 \qquad \text{and both} \qquad g_{xx} > 0 \text{ and } g_{yy} > 0 \qquad \text{(all at } (x_0, y_0)). \tag{2}$$

The critical point is neither a maximum nor a minum if $|D| < 0$, while the matter is undecided if $|D| = 0$. (For a function of κ variables, the requirements (1) and (2) would be replaced by requiring that the matrix analog of D be negative definite for a maximum and positive definite for a minimum.)

2 Integration

2.1 Basics

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + \text{Const}$$

$$\int a^x dx = \frac{a^x}{\ln a} + \text{Const}$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + \text{Const}$$

2.2 Integration by Parts

$$\begin{aligned} \int_{x_1}^{x_2} F(x)g(x)dx &= \Big[F(x)G(x)\Big]_{x=x_1}^{x=x_2} - \int_{x_1}^{x_2} f(x)G(x)dx \\ &= F(x_2)G(x_2) - F(x_1)G(x_1) - \int_{x_1}^{x_2} f(x)G(x)dx \end{aligned}$$

where $F(x) = \int_{-\infty}^{x} f(t)dt$ and $G(x) = \int_{-\infty}^{x} g(t)dt$.

3 Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$ then

$$F(x) = \int_a^x f(t)dt$$

is differentiable at every point x in $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

4 First Mean Value Theorem

If $f(x)$ and $f'(x) = df(x)/dx$ are continuous on [a,b] and $f'(x)$ is differentiable on (a, b), with derivative $f''(x) = d^2f(x)/dx^2$, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2$$

4.1 Linear approximation

If g, g' and g'' are continuous on the closed interval $[a, x]$ then we can make a linear approximation to $g(x)$:

$$g(x) \approx g(a) + g'(a)(x - a)$$

with the error of approximation bounded:

$$\left| g(x) - \Big(g(a) + g'(a)(x - a) \Big) \right| \leq \frac{1}{2} \max_{u \in [a,x]} |g''(u)|(x - a)^2$$

5 Taylor Series

The Taylor series for $f(x)$ at $x = a$ is given by:

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots$$

5.1 Common Taylor Series

Here are some common Taylor Series (about the point $a = 0$).

$$\begin{aligned}
\tfrac{b}{(1-x)} &= \sum_{k=0}^{\infty} bx^k && \text{for } |x| < 1 \\
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} && \text{for all } x \\
\ln(1 + x) &= \sum_{k=0}^{\infty} \frac{(-1)^{k-1}x^k}{k} && \text{for } -1 < x \leq 1
\end{aligned}$$

6 Binomial Theorem

If m is a positive integer then

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}$$

where

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}$$

is the binomial coefficient.

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