

Reconstructing Two-Dimensional Manifolds from  
Scattered Data:  
Motivation and Background

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## **Abstract**

In this report we introduce and motivate the problem of reconstructing shapes from partial information. An appropriate mathematical abstraction capturing the notion of a shape in three-dimensional space is a two-dimensional manifold. The concept of the topological type of a manifold plays an important role in reconstruction, and we present a synopsis of the pertinent definitions and results. We then discuss ways of representing two-dimensional manifolds. Finally, we focus on the specific problem of reconstructing a two-dimensional manifold from an unorganized collection of points assumed to be scattered on or about the manifold, and give a survey of previous work on this topic.

# 1 Introduction and Motivation

In very general terms, the class of problems we are interested in can be stated as follows: Given partial information of an unknown “target” shape, construct, to the extent possible, a representation of the shape. Reconstruction problems of this sort occur in diverse scientific and engineering application domains, including:

- *Surfaces from contours.* In many medical studies it is common to slice biological specimens into thin layers with a microtome. The outlines of the structures of interest are then manually digitized to create a stack of contours. The problem is to reconstruct the three-dimensional structures from the stacks of two-dimensional contours. Although this problem has received a good deal of attention, there remain severe limitations with current methods. Perhaps foremost among these is the difficulty in automatically dealing with branching structures [4, 21]. The motivation for reconstruction in this application is typical: the reconstructed surface generally requires less storage, and is therefore faster to transmit, process, display, etc.; metric properties such as surface area and volume are simpler to compute, the reconstructed surface is resolution independent, meaning that the model can be arbitrarily zoomed, rotated and otherwise transformed.
- *Scattered data interpolation.* Automobile bodies are most commonly designed by constructing a large scale model out of wood, clay, or fiberglass. Roughly speaking, the model is then digitized using an automated laser digitizer, and a spline representation is constructed to interpolate to the digitized data. Once in spline form, the surface can be efficiently edited and analyzed using CAD tools such as CATIA and PATRAN. The spline model can also be used to drive numerically controlled milling machines for the manufacture of casts, dies, and templates.

Unfortunately, the process is currently not as general as indicated. The reason is that the laser digitization systems operate by digitizing numerous curves lying on the surface of the object to be digitized. Each of the curves is reported as a sequence of points in three-dimensional space, but there is in general no automated correspondence between the points on one curve and the points on another. For relatively planar rectangular objects, such as the hood of an automobile, the individual sequences can be stored in a two-dimensional array to produce a rectangular grid of points, as was mentioned above. More complex objects, such as a milk pitcher with a handle, cannot be covered with a single rectangular grid, so more sophisticated algorithms are needed for establishing a two-dimensional network of points. The digitization of objects using manually

positioned digitizers is also made easier if explicit organization does not have to be manually specified.

- *Curve and surface sketching.* Popular curve design systems, such as those found in mechanical CAD and typographic systems, require the user to interact with the curve through a set of “handles” or “control points”. This is the case, for instance, for systems based on Bézier or B-spline curves [6]. A more intuitive method that has seen some investigation is to have the system construct a Bézier or B-spline representation from a “sketch” provided by the user. In this style of interaction the system queries the mouse while the user traces out the desired shape, resulting in a sequence of points deemed close to the target curve. The system then automatically smoothes the sequence and invokes a fitting procedure to produce a spline representation (cf. Schneider [18], Plass and Stone [16]). These procedures are based on the assumption that the input is ordered along the curve, meaning that the user is required to sweep out the curve in one continuous motion. Unfortunately, most users would prefer to sketch by stroking repeatedly back and forth over the target shape. Algorithms are therefore needed to compute ordering information from initially unordered or partially ordered data.

With the advent of sophisticated input devices such as the “Data Glove” (a sensor glove capable of reporting the position and orientation of the users hand and fingers [7]), we can imagine designing complex three-dimensional objects by sketching in three-dimensions. The user would wear a Data Glove on each hand, and indicate the target shape by moving his hands in a region around the desired surface. During this time the system could record a large number of positions of the hands, thereby building up a “cloud” of points near the desired surface. The difficult problem then is to construct a surface representation faithful to this largely unordered collection of points.

- *Fitting of object boundaries in images.* An important problem in image analysis is the detection and description of object boundaries. Application of an edge detector to the image results in an unstructured collection of pixels thought to be on or near the boundaries. The edge pixels then have to be partitioned into subsets corresponding to individual objects. For later analysis it is desirable to summarize the potentially large number of edge pixels associated with an object by a continuous curve. This approach has for example been used by Banfield and Raftery [2, 1] for detection and description of ice floes on satellite images.

Another closely related application is quantifying the shape of biological structures in CT or MRI images. One way to accomplish this is to use a low level edge detector to identify voxels, possibly tagged with tangent planes, thought to be

near the boundary of the structure, then to use a reconstruction algorithm to produce a continuous, closed surface representation. In addition to the extreme compression factor achievable by reconstruction, the measurement of surface areas and volumes is important here for diagnosis and treatment planning.

Diverse problems such as those listed above can be unified into a single problem space. The resulting space possesses a number of independent dimensions, including:

- *The structure of the input data.* As the applications above indicate, the input to reconstruction problems can vary widely: On one extreme, the data might be an unstructured collection of points. On the other extreme we might be given, in addition to the points, the edges and faces of a mesh approximating the surface.
- *The amount of noise present in the data.* Interpolation problems assume that no noise is present, whereas there is generally a non-negligible noise level in problems such as fitting object boundaries in images and curve or surface sketching.
- *The density of the data.* The sketching and image fitting problems typically assume relatively high density data distributed around the target shape, whereas many interpolation problems are characterized by very sparsely distributed data.
- *The uniformity of the data.* The data may be distributed with varying degrees of uniformity over the target shape.
- *Knowledge of the solution space.* The reconstructed model must be chosen from some class of possible models. We call this class the solution space of the problem. For instance, in the surfaces from contours problem the reconstructed model is intended to represent the boundary of a three-dimensional biological volume, meaning that the reconstructed surface should be chosen from the class of compact two-dimensional manifolds without boundary. Other problems, such as the fitting of boundaries in images, require curves (i.e., one-dimensional manifolds) as output. Applications requiring higher dimensional manifolds or manifolds with boundary also exist. Different applications additionally place different requirements on the continuity or smoothness of the reconstructed model. Whereas positional continuity may be sufficient for medical imaging, many engineering applications require models that are twice differentiable (i.e., of class  $C^2$ ).

A reconstruction problem can therefore be denoted by its “location” in the problem space. The purpose of this paper is to provide some background relevant to all

reconstruction problems involving two-dimensional shapes in three-dimensional space, and to survey previous work on the specific problem of reconstructing a shape from an unorganized collection of points.

The remainder of the paper is organized as follows: In section 2 we introduce two-dimensional manifolds and give examples. Two-dimensional manifolds can be classified according to their *topological type*, a property that plays an important role in reconstruction. We provide a synopsis of relevant mathematical definitions and results. In section 3 we discuss ways of representing two-dimensional manifolds, with emphasis on representations that appear to be useful for reconstruction. Finally, in section 4 we focus more specifically on the problem of reconstructing a two-dimensional shape from an unorganized collection of data points, and present a survey of previous work on this topic.

## 2 Topology

For the development and analysis of reconstruction procedures it is helpful to have some understanding of the theory of two-dimensional manifolds. In particular, it is important to appreciate the concept of the *topological type* of a manifold: a torus is fundamentally different from a sphere, (or from a piece of the plane), and a reconstruction procedure that smoothly deforms a sphere will not be successful in approximating a surface that is shaped like a doughnut. On the other hand, there really is no fundamental difference between a cube, a tetrahedron, and a sphere.

An important question to ask about a reconstruction procedure is its scope: can it deal with manifolds of arbitrary topological type, or only with a restricted class, such as manifolds that look like a piece of the plane? The ability of constraining a reconstruction procedure to produce an estimate of pre-specified topological type is also important. Often the topological type of the target is known in advance, and the procedure should be able to use this knowledge for resolving ambiguities.

### 2.1 Manifolds

The following definitions and facts about manifolds can be found in standard texts on algebraic topology, for example the book by Massey [12].

**Def 2.1** *A Hausdorff space  $M$  is called a  $n$ -dimensional manifold if for every  $\mathbf{x} \in M$  there is an open neighborhood  $W$  of  $\mathbf{x}$  and a homeomorphism  $\phi_W$  mapping  $W$  onto the open ball  $B^n = \{\mathbf{x} \in R^n : \|\mathbf{x}\| < 1\}$  in  $R^n$ . A tuple  $(W, \phi_W)$  is called a chart. A collection of charts covering  $M$  is called an atlas for  $M$ .*

Recall that a topological space  $M$  is called a Hausdorff space if for any two points of  $M$  there are disjoint open sets, each containing just one of the two points. A mapping  $f$  is called a homeomorphism if it is one-to-one and both  $f$  and  $f^{-1}$  are continuous.

**Note:** If  $M$  is compact, it has an atlas consisting of finitely many charts.

**Examples:**

- The two-sphere  $S^2 = \{\mathbf{x} \in R^3 : \|\mathbf{x}\| = 1\}$  is a compact two-dimensional manifold.
- The torus  $T = \{\mathbf{x} \in R^3 : (\sqrt{(x_1^2 + x_2^2)} - 2)^2 + x_3^2 = 1\}$  is a compact two-dimensional manifold.
- The Moebius strip obtained by identifying points  $(0, x_2)$  and  $(1, 1 - x_2)$ ,  $0 < x_2 < 1$  of the square  $[0, 1] \times (0, 1)$  is a non-compact two-dimensional manifold. We can construct a physical realization of a Moebius strip by taking a long, narrow strip of paper and gluing the ends together with a half twist. The fact that the  $x_2$ -interval is open is important. If the interval was closed, points with  $x_2 = 0$  or  $x_2 = 1$  would not have open neighborhoods homeomorphic to  $B$ , and we would have a compact manifold with boundary (see below).
- The projective plane obtained by identifying diametrically opposite points of the two-sphere  $S^2$  is a compact two-dimensional manifold that cannot be embedded into  $R^3$ ; that is, there is no subset of  $R^3$  homeomorphic to a projective plane.

**Def 2.2** A Hausdorff space  $M$  is called a  $n$ -dimensional manifold with boundary if every  $\mathbf{x} \in M$  has an open neighborhood  $W$  that is either homeomorphic to  $B^n$  or to the set  $B_+^n = \{\mathbf{x} \in B^n : x_1 \geq 0\}$ . The set of points that have a neighborhood homeomorphic to  $B^n$  is called the interior of  $M$ . The set of points  $\mathbf{x}$  that have an open neighborhood  $V$  such that there exists a homeomorphism  $h : V \rightarrow B_+^n$  with  $h(\mathbf{x}) = \mathbf{0}$  is called the boundary of  $M$ .

**Example:**  $M = \{\mathbf{x} \in R^3 : \|\mathbf{x}\| = 1, x_3 \geq 0\}$  is a two-dimensional manifold with boundary. The boundary is the circle  $x_3 = 0, x_1^2 + x_2^2 = 1$ .

It can be shown that the boundary of a  $n$ -dimensional manifold with boundary is an  $n - 1$ -dimensional manifold (without boundary).

**Def 2.3** A  $n$ -dimensional manifold  $M$  is called smooth if for any two charts  $(U, \phi_U)$  and  $(V, \phi_V)$  with  $U \cap V \neq \emptyset$  the mapping  $\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \rightarrow \phi_V(U \cap V)$  is a diffeomorphism.

Recall that a mapping  $f$  is called a diffeomorphism if it is a homeomorphism and both  $f$  and  $f^{-1}$  are smooth (infinitely often differentiable).

## 2.2 Surfaces

We have so far been intentionally avoiding the term “surface”. While it is common to talk about “the surface of a sphere”, many so-called surface reconstruction procedures can only cope with a restricted class of two-dimensional manifolds, namely those that can be represented as graphs of functions over the plane; see Lancaster and Salkauskas [11] or Bolle and Vemuri [3] for examples. (The graph of a function  $f : R^2 \rightarrow R$  is defined as  $G(f) = \{\mathbf{x} \in R^3 : x_3 = f(x_1, x_2)\}$ ). Note that the graph of a continuous function  $f : R^2 \rightarrow R$  is a two-dimensional manifold in  $R^2 \times R^1$ . However, not every two-dimensional manifold can be represented as the graph of a continuous function on  $R^2$  or a subset thereof.

A two-dimensional manifold can consist of several separated components, each of them a two-dimensional manifold in its own right. Also, a manifold might or might not be compact. In the following, the term *surface* will be used in a specific technical sense:

**Def 2.4** *A compact, connected two-dimensional manifold with boundary is called a surface. A compact, connected two-dimensional manifold without boundary is called a closed surface.*

If we want to emphasize that a surface has non-empty boundary, we use the term *bordered surface*.

## 2.3 Simplicial surfaces

Roughly speaking, a *simplicial surface* is a surface consisting of planar triangular facets pasted together along their edges. Simplicial surfaces are important for several reasons. First, one can show that any surface is homeomorphic to a simplicial surface. Second, the topological type of a simplicial surface can be determined by purely combinatorial means, and third, it is easy to create simplicial surfaces of any desired topological type. The following presentation of simplicial surfaces follows Spanier [22] and Hudson [10].

**Def 2.5** *A (finite) simplicial complex  $K$  consists of a finite set  $V$ , the vertices of  $K$ , together with a set  $S$  of non-empty subsets of  $V$ , called the simplexes of  $K$ , such that*

1. *any set consisting of exactly one vertex is a simplex*
2. *every non-empty subset of a simplex is again a simplex.*



A simplex  $s$  containing exactly  $q + 1$  vertices is called a  $q$ -simplex and we say that the *dimension* of  $s$  is  $q$  and write  $\dim s = q$ . The *dimension* of  $K$ , written  $\dim K$ , is the maximum dimension of its simplexes. The  $q$ -skeleton of  $K$  is the simplicial complex  $K^q$  consisting of the simplexes of  $K$  of dimension at most  $q$ . We identify each 0-simplex with its single element, then  $K^0 = V$ . A (*proper*) *face* of a simplex  $s$  is a non-empty (proper) subset of  $s$ ; and we write  $s' \leq s$  to denote that  $s'$  is a face of  $s$ .

**Example:** Recall that a *simple graph* is a graph with the property that any two vertices are joined by at most one edge, and which contains no loops (edges with only one endpoint). Thus, a simplicial complex of dimension one is just a simple graph.

A *simplicial map*,  $\phi : K_1 \rightarrow K_2$  is a function from the vertices of  $K_1$  to the vertices of  $K_2$ , such that for any simplex  $s$  of  $K_1$ ,  $\phi(s)$  is a simplex of  $K_2$ . Two simplicial complexes are *isomorphic* if there is a simplicial map which is a bijection.

Associated to each simplicial complex  $K$  is a topological space  $|K|$ , called its *topological realization*:

**Def 2.6** Let  $K$  be a simplicial complex. Identify the vertices  $\{v_1, v_2, \dots, v_n\}$  of  $K$  with the standard basis vectors of  $R^n$  and for each simplex  $s$  let  $|s|$  denote the convex hull of its vertices. Then  $|K| = \cup_{s \in K} |s|$ .

Here we are concerned with simplicial complexes whose topological realizations are surfaces:

**Def 2.7** Let  $K$  be a simplicial complex. If  $|K|$  is a surface, then  $|K|$  is called a *simplicial surface*.

It is useful to have a combinatorial criterion for a simplicial complex to be a surface. One such criterion is based on the structure of the neighborhoods of simplexes. Let  $s$  be a simplex of  $K$ . Then

$$\text{star}(s; K) = \{s' \in K : s \leq s'\}$$

$$\overline{\text{star}}(s; K) = \{s' \in K : s' \text{ is a face of an element of } \text{star}(s; K)\}$$

$$\text{link}(s; K) = \overline{\text{star}}(s; K) \setminus \text{star}(s; K).$$

Notice that  $\text{link}(s; K)$  and  $\overline{\text{star}}(s; K)$  are simplicial complexes. The subset  $|\overline{\text{star}}(s; K)| \subset |K|$  is the smallest (closed) neighborhood of  $|s|$  in  $|K|$  which is a union of simplexes;  $|\text{link}(s; K)|$  is the boundary of  $|\overline{\text{star}}(s; K)|$  and  $|\overline{\text{star}}(s; K)| \setminus |\text{link}(s; K)|$  is an open neighborhood of  $|s|$ .

**Prop 2.1** *Let  $K$  be a simplicial complex of dimension 2. Then  $K$  is a simplicial surface if and only if for every vertex  $v$  of  $K$  the space  $|\text{link}(v; K)|$  is homeomorphic to either a circle (in which case  $v$  is an interior vertex of  $|K|$ ) or a closed interval (in which case  $v$  is a boundary vertex of  $|K|$ ).*

Let  $K$  be a simplicial surface. Its *boundary*  $\partial K$  is the simplicial complex consisting of the 1-simplexes of  $K$  that are each a face of only one 2-simplex, together with the vertices of such 1-simplexes. It is not difficult to see that the topological space  $|\partial K|$  coincides with the boundary of the (topological) surface  $|K|$ .

**Def 2.8** *Let  $M$  be a surface. A triangulation of  $M$  is a homeomorphism  $\phi : |K| \rightarrow M$ . The images under  $\phi$  of the 2-simplexes, 1-simplexes, and 0-simplexes are called triangles, edges, and vertices.*

**Prop 2.2** *Any surface can be triangulated.*

## 2.4 Classification of closed surfaces

Reconstruction of compact, connected one-dimensional manifolds is greatly simplified by the fact that there are only two types — those that are homeomorphic to a circle and those that are homeomorphic to a closed interval. For surfaces the situation is more complicated.

To state the main results on surface classification, we have to introduce the notion of a *connected sum* of surfaces. Let  $M_1, M_2$  be disjoint surfaces. Their connected sum  $M_1 \sharp M_2$  is formed by cutting an open disc out of each surface and then gluing the two surfaces together along the boundaries of the holes.

More precisely, let  $\mathbf{x}_1, \mathbf{x}_2$  be interior points of  $M_1$  and  $M_2$ , respectively. Let  $N_1, N_2$  be (closed) neighborhoods of  $\mathbf{x}_1, \mathbf{x}_2$  homeomorphic to the closed unit ball  $\overline{B^2}$ , and let  $\phi_1 : N_1 \rightarrow \overline{B^2}$  and  $\phi_2 : N_2 \rightarrow \overline{B^2}$  denote the corresponding homeomorphisms. Set  $U_1 = M_1 \setminus \overset{\circ}{N}_1$  and  $U_2 = M_2 \setminus \overset{\circ}{N}_2$ . Define an equivalence relation “ $\sim$ ” on  $U_1 \cup U_2$  by  $\mathbf{x} \sim \mathbf{y}$  if  $\phi_1(\mathbf{x}) = \phi_2(\mathbf{y})$ . The connected sum  $M_1 \sharp M_2$  is the set of equivalence classes  $U_1 \cup U_2 \text{ mod } \sim$ , endowed with the finest topology that makes the natural projection of  $U_1 \cup U_2$  onto  $M_1 \sharp M_2$  continuous.

It can be shown that  $M_1 \sharp M_2$  is again a surface, and that its topological type does not depend on the location of the holes.

**Note:** The connected sum of any surface  $M$  with the two-sphere  $S^2$  is homeomorphic to  $M$  itself.

**Prop 2.3 (Classification theorem for closed surfaces):** *Any closed surface is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes.*

**Note:** It is important that  $M$  be compact and without boundary. The former rules out surfaces like the open disc  $\{\mathbf{x} \in R^3 : x_1^2 + x_2^2 < 1, x_3 = 0\}$ ; the latter rules out the corresponding closed disc.

There remains the practical problem of finding the topological type of a surface. This is usually done by first constructing a triangulation of the surface. The topological type can then be determined from the triangulation by purely combinatorial means.

### 2.4.1 Euler characteristic

Given a triangulated surface, we can compute its *Euler characteristic*:

**Def 2.9** *Let  $v, e, f$  denote the number of vertices, edges, and triangles of a triangulated surface  $M$ . Then the integer*

$$\chi(M) = v - e + f$$

*is called the Euler characteristic of  $M$ .*

It can be shown that the Euler characteristic of  $M$  is a property of  $M$  and does not depend on the particular triangulation. Homeomorphic surfaces have the same Euler characteristic.

It is easy to see that the Euler characteristic of the two-sphere is  $\chi(S^2) = 2$ :  $S^2$  is homeomorphic to a tetrahedron, which has 4 triangles, 4 vertices, and 6 edges.

To find the Euler characteristic of a torus, note that a torus can be obtained by partitioning the unit square into 9 sub-squares, each one split into two triangles, and then identifying corresponding edges on the left and right sides of the square, and on the bottom and top of the square. This leaves 9 vertices, 27 edges, and 18 triangles, showing that the torus has Euler characteristic 0.

Using a similar argument as for the torus, it can be shown that the projective plane has Euler characteristic 1.

There is a simple relationship between the Euler characteristic of two surfaces  $M_1$  and  $M_2$ , and the Euler characteristic of their connected sum  $M_1 \sharp M_2$ . Without loss of generality we may assume that  $M_1$  and  $M_2$  are triangulated. To form the connected sum, we cut out the interiors of two triangles, one on each surface, and then identify pairs of edges of these triangles, one from each surface. This reduces the number of vertices by 3, the number of edges by 3, and the number of triangles by 2:

**Prop 2.4**  $\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - 2$ .

Thus the Euler characteristic of the connected sum of  $n$  two-spheres is 2 (obviously), the Euler characteristic of the connected sum of  $n$  Tori is  $2 - 2n$ , and the Euler characteristic of the connected sum of  $n$  projective planes is  $2 - n$ . Therefore a torus has the same Euler characteristic as the connected sum of two projective planes. This shows that the topological type of a surface cannot be inferred from the Euler characteristic alone. We need to consider another topological property of surfaces, namely *orientability*.

### 2.4.2 Orientability

We will first define orientability for simplicial surfaces. Let  $|K|$  be a simplicial surface, with 2-simplexes  $s_1, \dots, s_n$ . A simplex is oriented by specifying an ordering of its vertices. Two orderings are called equivalent if one can be obtained from the other by an even permutation of the vertices. By orienting a 2-simplex  $s$  we define the meaning of “walking clockwise around the edges of  $s$ ”.  $|K|$  is called orientable if its 2-simplexes can be oriented in a consistent fashion. Roughly speaking, what we mean by a consistent orientation of 2-simplexes is that the definition of “clockwise” does not change as we cross the edge between one 2-simplex and an adjacent 2-simplex.

Here is a more precise definition. Suppose the 1-simplex  $e$  is a face of both  $s_i$  and  $s_k$ . The orientation of  $s_i$  induces an orientation of  $e$ , and so does the orientation of  $s_k$ . For the orientation of  $s_i$  and  $s_k$  to be consistent, the two orientations induced on  $e$  have to be opposite. This definition of “consistent orientation” immediately leads to an algorithm for deciding whether a simplicial surface is orientable or not: Define a graph whose vertices are the 2-simplexes  $s_1, \dots, s_n$ . Two vertices are connected by an edge if the corresponding 2-simplexes share a 1-simplex as a face. Construct a spanning tree of this graph and root the tree at some arbitrary vertex. Traverse the tree, starting at the root, orienting each 2-simplex consistent with the orientation of its parent. Finally, loop over all the 1-simplexes and check whether each of them is assigned different orientations by the 2-simplexes of which it is a face. If this is the case, the surface is orientable, otherwise it is not.

A surface is called *orientable* if it has an orientable triangulation. It is a nontrivial fact that orientability is indeed a property of the surface and does not depend on the triangulation. Like the Euler characteristic, orientability is a topological invariant: homeomorphic surfaces are either both orientable or both non-orientable.

It is easy to see that the connected sum  $M_1 \sharp M_2$  of two surfaces is orientable if and only if both  $M_1$  and  $M_2$  are orientable. While the two-sphere  $S^2$  and the torus are

orientable, the projective plane is not. Indeed there is a subset of the projective plane that is homeomorphic to a Moebius strip, and a Moebius strip is non-orientable. Because Euler characteristic as well as orientability are topological invariants, and because of the classification theorem for closed surfaces, we have

**Prop 2.5** *Two closed surfaces  $M_1$  and  $M_2$  are homeomorphic if and only if their Euler characteristics are equal and they are both orientable or both non-orientable.*

## 2.5 Classification of bordered surfaces

Next we will consider classification of bordered surfaces. Let  $M$  be a surface with  $k$  boundary components. Each boundary component is homeomorphic to a circle. If we take  $k$  closed discs and glue the boundary of the  $i$ -th disc to the  $i$ -th boundary component of  $M$ , we obtain a closed surface  $M^*$ .

**Prop 2.6 (Classification theorem for bordered surfaces):** *Two bordered surfaces  $M_1$  and  $M_2$  are homeomorphic if and only if they have the same number of boundary components, and  $M_1^*$  and  $M_2^*$  are homeomorphic.*

**Note:** If we take a (non-orientable) Moebius strip, which has a single boundary component, and glue a disc to the boundary, we obtain a projective plane, which is also non-orientable.  $M^*$  is orientable if and only if  $M$  is orientable.

As  $\chi(M^*) = \chi(M) + k$ , where  $k$  is the number of boundary components, we have:

**Prop 2.7** *Two bordered surfaces  $M_1$  and  $M_2$  are homeomorphic if they have the same number of boundary components, the same Euler characteristic, and are both orientable or both non-orientable.*

## 3 Representation of surfaces

A crucial issue in the development of methods for manifold reconstruction is the choice of a representation for the reconstructed manifold. Ideally, a representation should allow us to represent manifolds of arbitrary topology. While this requirement might be too stringent and not absolutely necessary (suppose we only want to reconstruct manifolds homeomorphic to  $S^2$ ), at the very least we ought to be able to determine the scope of the representation, i.e. the topological types that it can actually represent. We also would like to be able to determine the topological type of a reconstruction. Without this ability it would clearly be hard to place constraints on the topological type.

There are two kinds of representations that have been used in the context of manifold reconstruction, *implicit representation* and *parametric representation*.

### 3.1 Implicit representation

Implicit representation is based on the fact that the zero-set  $Z(f) = f^{-1}(0)$  of a function  $f : R^3 \rightarrow R$  that is smooth ( $C^\infty$ ) in a neighborhood of  $f^{-1}(0)$  and for which 0 is a regular value is a two-dimensional manifold in  $R^3$ . This is a direct result of the implicit function theorem. Recall that a value  $y$  of a differentiable function  $f(x)$  is called *regular* if the derivative of  $f(x)$  has full rank for all  $x \in f^{-1}(y)$ .

For example, the two-sphere  $S^2$  can be defined as the zero-set of the function  $f(\mathbf{x}) = \|\mathbf{x}\|^2 - 1$ . More generally, any smooth, closed surface  $M \subset R^3$  can be implicitly represented as the zero-set of the associated *signed distance function*. Any such surface divides  $R^3$  into two components (“inside” and “outside”), and the signed distance function is defined as  $f(\mathbf{x}) = s(\mathbf{x}) \cdot d^2(\mathbf{x}, M)$ , where  $s(\mathbf{x}) = 1$  if  $\mathbf{x}$  is inside,  $s(\mathbf{x}) = -1$  if  $\mathbf{x}$  is outside, and  $d(\mathbf{x}, M)$  is the distance between  $\mathbf{x}$  and its closest point in  $M$ . The signed distance function is smooth in a neighborhood of  $M$ , and 0 obviously is a regular value.

Note, however, that not every surface, even if it is smooth, can be implicitly represented as the zero-set of a smooth function  $f$  whose domain is  $R^3$ . It is easy to see that the zero-set of any smooth function on  $R^3$  for which 0 is a regular value divides  $R^3$  into at least two separated components. Therefore a surface homeomorphic to a disc, or to a torus with a hole cut into it, cannot be represented in this way.

To implicitly represent bordered surfaces, the domain  $D$  of  $f$  cannot be  $R^3$  — it has to be a smooth, compact submanifold of  $R^3$  with smooth boundary. Moreover, 0 has to be a regular value of both  $f$  and  $f|_{\partial D}$  ( $f$  restricted to the boundary of  $D$ ). It can be shown that in this case  $Z(f)$  is a two-dimensional manifold (not necessarily connected), and the boundary of  $Z(f)$  is exactly the intersection of  $Z(f)$  with the boundary of  $D$  (see Milnor [13] for further explanation). Obviously, the domain  $D$  as well as the function  $f$  need to be considered part of the representation.

### 3.2 Parametric representation

An alternative to implicit representation is *parametric representation*: represent  $M$  by a pair  $(\Lambda, \mathbf{f})$ , where the *parameter space*  $\Lambda$  is itself a surface, and the mapping  $\mathbf{f} : \Lambda \rightarrow R^3$  is continuous, one-to-one, and  $\mathbf{f}^{-1}$  is continuous on the range  $\mathbf{f}(\Lambda)$  of  $\mathbf{f}$ . A mapping  $\mathbf{f}$  with these properties is called a *topological embedding*, and it is trivial to verify that  $\mathbf{f}(\Lambda) \subset R^3$  is indeed a surface homeomorphic to  $\Lambda$ . We use the

term parametric to stress the analogy with parameterized curves: every point on the surface is uniquely associated with a parameter value  $\lambda \in \Lambda$ .

The following examples illustrate the idea:

**Example 1:** Let  $\Lambda$  be a closed disc in  $R^2$  (or, more generally, any compact, connected region with smooth boundary), and set  $f_1(\lambda_1, \lambda_2) = \lambda_1$ ,  $f_2(\lambda_1, \lambda_2) = \lambda_2$ , and  $f_3(\lambda_1, \lambda_2) = h(\lambda_1, \lambda_2)$  for some continuous function  $h : \Lambda \rightarrow R$ . Then  $\mathbf{f}$  is a topological embedding, and the graph of  $h$ ,  $G(h) = \{(\lambda_1, \lambda_2, h(\lambda_1, \lambda_2))\}$ , is a surface homeomorphic to  $\Lambda$ . Note, however, that *not* every surface homeomorphic to a closed disc is the graph of a continuous function over a planar domain. For a counterexample consider  $M = \{\mathbf{x} \in S^2 : x_1 \geq -0.5\}$ . Reconstruction procedures that represent the reconstructed manifold as the graph of a function over a planar domain are thus quite limited, as they cannot even deal with all manifolds homeomorphic to a disc.

**Example 2:** Let  $\Lambda$  be the two-sphere  $S^2$ , and let  $h(\theta, \phi) : S^2 \rightarrow R$  denote a continuous function with  $h(\theta, \phi) > 0$  everywhere. Then the graph of  $h$ , i.e. the set of points in  $R^3$  with polar coordinates  $(\theta, \phi, h(\theta, \phi))$  is an closed surface homeomorphic to  $S^2$ . An obvious question is under which conditions a closed surface  $M$  can be represented as the graph of a function over the sphere. To answer this question, we have to introduce the concept of a *star-shaped set*: A set  $S$  is called star-shaped if there is a point  $\mathbf{c}$  such that for every  $\mathbf{x} \in S$  the line segment  $(1 - \lambda)\mathbf{x} + \lambda\mathbf{c}, 0 \leq \lambda \leq 1$ , lies entirely within  $S$ .

It can be shown that a closed surface  $M$  is the graph of a function over the sphere exactly if its inside is star-shaped relative to  $\mathbf{c} = \mathbf{0}$  and, moreover, each line segment  $(1 - \lambda)\mathbf{x} + \lambda\mathbf{c}, 0 \leq \lambda \leq 1$  intersects  $M$  exactly once, namely for  $\lambda = 1$ . If  $M$  is star-shaped relative to some point  $\mathbf{c} \neq \mathbf{0}$  and has the intersection property, we can represent it as a translation of the graph of a function over the sphere. Even allowing for translation, reconstruction methods that represent the reconstructed manifold as the graph of a function over the sphere can handle only a subset of the surfaces that are homeomorphic to a sphere.

**Example 3:** Choose  $\Lambda$  to be a closed disc, and let  $\mathbf{f} : \Lambda \rightarrow R^3$  be one-to-one and differentiable, with the additional property that the derivative of  $\mathbf{f}$  has rank 2 everywhere. The conditions on  $f$  guarantee that  $f$  is a topological embedding, and therefore  $\mathbf{f}(\Lambda)$  is a surface homeomorphic to  $\Lambda$ . Moreover, any smooth surface homeomorphic to a closed disc can be represented in this way.

Note that all parametric representations mentioned in the examples are limited in the topological type of the surfaces they can represent. To parametrically represent

a surface of some arbitrary type, we first of all have to construct a parameter domain  $\Lambda$  of this type. Second, we need a way of representing topological embeddings of  $\Lambda$  into  $R^3$ . How to do this will be the subject of a future report.

## 4 Previous work on surface reconstruction from unorganized data

In this section we consider the problem of reconstructing a surface  $M \subset R^3$  from an unorganized collection  $\mathbf{x}_1, \dots, \mathbf{x}_n$  of points assumed to be scattered on or about the surface.

Reconstruction methods can be classified according to the way in which they represent the manifold. We will first discuss methods using implicit representation.

### 4.1 Reconstruction methods using implicit representation

Several authors (Pratt [17], Taubin [23], and Moore and Warren [14], among others) have suggested reconstruction methods using implicit representations: find a smooth function  $f : R^3 \rightarrow R$  such that the data points  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are close to the zero-set  $Z(f)$ . Their methods differ in the way in which  $f$  is found.

Pratt and Taubin take  $f$  to be a polynomial  $p_{\mathbf{a}}(\mathbf{x})$  in three variables, with coefficient vector  $\mathbf{a}$ . The goal is to find  $\mathbf{a}$  minimizing  $\sum d^2(\mathbf{x}_i, Z(p_{\mathbf{a}}))$ , the sum of squared distances from the data points to the zero-set of  $p_{\mathbf{a}}$ , subject to the constraint that 0 be a regular value of  $p_{\mathbf{a}}$ . Finding the optimal  $\mathbf{a}$  is a difficult problem, and one has to be satisfied with an approximate solution. The condition that 0 be a regular value of  $p_{\mathbf{a}}$  is hard to verify, and in practice one only requires that  $\nabla p_{\mathbf{a}} \neq 0$  at the data points  $\mathbf{x}_1, \dots, \mathbf{x}_n$ .

There are several other problems with this approach. First, there is the problem of finding the appropriate domain for  $p_{\mathbf{a}}$ . The zero-set of  $p_{\mathbf{a}}$  will often have parts that are not supported by any data, and these parts have to be removed by restricting the domain of  $p_{\mathbf{a}}$ . It is not obvious how this could be accomplished.

Second, it is not known how one could force  $Z(p_{\mathbf{a}}|D)$ , the zero-set of the polynomial on the domain  $D$ , to have a pre-specified topology. There are many scenarios where the topological type of the surface to be reconstructed would be known, and one should be able to make use of this knowledge in the reconstruction process. Even determining the topological type of  $Z(p_{\mathbf{a}}|D)$  is difficult.

The report by Moore and Warren [14]) contains several interesting variations on the theme of implicit reconstruction. Instead of insisting on representing the entire man-



ifold by the zero-set of a single polynomial (of potentially high order), they construct a piecewise representation by low order polynomials. The domain of the initial fit is a tetrahedron  $T_0$  containing all the data points. If the total squared distance between the data points and the initial fit (or an approximation thereof) is too large, the tetrahedron is subdivided, and the fitting procedure is recursively applied to the sub-tetrahedra so obtained. The end result is a collection of tetrahedra  $T_1, \dots, T_k$  and associated functions  $f_1, \dots, f_k$ . The estimate of the piece of the surface passing through  $T_i$  is the zero-set of  $f_i$ . In general, however, these pieces will not fit together continuously.

To achieve continuity, Moore and Warren use a technique they call *free form blending*. First, the decomposition of  $T_0$  into tetrahedra is expanded into a valid triangulation. Every vertex  $\mathbf{v}$  of this triangulation is shared by some set of tetrahedra  $T_1, \dots, T_k$ , each one with its associated function  $f_k$ . A new function value for the vertex  $\mathbf{v}$  is computed as weighted average of the values of the  $f_k$ , with weights depending on the number of data points on which  $f_k$  was based. (If desired, derivatives can be treated the same way). This results in a single function value for each vertex. Finally these function values are interpolated over  $T_0$ , and the estimate for the underlying surface is taken to be the zero set of the interpolant.

As an alternative to finding a polynomial whose zero-set approximates the underlying surface, Moore and Warren suggest using an estimate of the signed distance function. If the domain  $D$  is chosen such that the surface  $M$  divides  $D$  into two components, then the signed distance function is well defined. Compared to a polynomial it has the advantage that its zero-set is guaranteed to have no spurious parts. Moore and Warren deal with the case where the domain  $D$  is a tetrahedron and the surface divides  $D$  into two connected subsets. However, much more complex scenarios could occur, and it is not obvious how to extend their approach.

## 4.2 Reconstruction methods using parametric representation

There is a vast literature on surface reconstruction restricted to the special case of surfaces that are graphs of functions over a planar domain. We shall not even attempt a survey here, but instead refer the reader to the book by Lancaster and Salkauskas [11] and the survey by Bolle and Vemuri [3].

Schudy and Ballard [19, 20] consider reconstruction of surfaces that are graphs of functions over the sphere. Their motivating application is reconstruction of the surface of the human heart from points  $\mathbf{x}_1, \dots, \mathbf{x}_n$  on the heart's surface. They first transform the points into polar coordinates  $(\theta_i, \phi_i, r_i)$  with respect to a suitably chosen center  $\mathbf{c}$ . (It is not quite clear how the center is found). The problem then is reduced to that of estimating a function  $r(\theta, \phi)$  from data  $\{(\theta_i, \phi_i, r_i), i = 1, \dots, n\}$ .

Schudy and Ballard represent  $r(\theta, \phi)$  as a linear combination of spherical harmonics:

$$r(\theta, \phi) = \sum_{j=1}^k a_j h_j(\theta, \phi)$$

and estimate the coefficients  $a_j$  by least squares:

$$\hat{\mathbf{a}} = \operatorname{argmin}_{\mathbf{a}} \sum_{i=1}^n (r_i - \sum_j a_j h_j(\theta_i, \phi_i))^2.$$

The exact definition of spherical harmonics is not important here. The important facts are that spherical harmonics are smooth functions over  $S^2$ , they can be arranged in increasing order of spatial complexity (the higher the index, the wigglier the function), and they form a basis of the Hilbert space of  $L_2$ -functions over the sphere. The latter property implies that any “reasonable” function over the sphere can be well approximated by spherical harmonics.

Instead of expanding the radius  $r(\theta, \phi)$  into spherical harmonics, one could use other collections of functions as a basis for the expansion. Several suggestions are presented by Foley [8] and Nielson et al [15], who discuss interpolation and approximation of functions over the sphere.

A reconstruction problem very similar to the one treated by Schudy and Ballard is discussed by Brinkley [5], who proposes a way of incorporating prior knowledge about the shape of the surface, i.e. the function  $r(\theta, \phi)$ , into the reconstruction process.

Hastie and Stuetzle [9] and Vemuri [25, 24] discuss reconstruction of surfaces by a topological embedding  $\mathbf{f}(\lambda)$  of a planar region  $\Lambda$  into  $R^3$ . The data points are considered to be (possibly noisy) observations of  $\mathbf{f}$ :

$$\mathbf{x}_i = \mathbf{f}(\lambda_i) + \epsilon_i,$$

where  $\lambda_i$  is the parameter vector corresponding to the  $i$ -th observation and  $\epsilon_i \in R^3$  is a noise component.

If the  $\lambda_i$  were known, the problem of estimating  $\mathbf{f}$  could be decomposed into three function estimation problems: Estimate the  $j$ -th coordinate function  $f_j$  from the dataset  $(\lambda_{i1}, \lambda_{i2}, x_{ij}), i = 1, \dots, n$ . This is the case considered by Vemuri. He assumes that the data points are obtained by a laser scanner and are thus naturally arranged in a rectangular grid  $\{\mathbf{x}_{ij}, i = 1, \dots, l, j = 1, \dots, m\}$ . The parameter vectors are arranged in a corresponding planar grid  $\{\lambda_{ij}\}$ . The spacing of the parameter grid is a two-dimensional generalization of the chord-length parameterization. The distance

between two vertical grid lines is taken to be proportional to the average distance between the corresponding data points:

$$\lambda_{i,j+1} - \lambda_{i,j} \sim \frac{1}{l} \sum_{i=1}^l \|\mathbf{x}_{i,j+1} - \mathbf{x}_{i,j}\|$$

with the proportionality factor chosen such that  $\lambda_{i1} = 0$  and  $\lambda_{im} = 1$ . The distance between horizontal grid lines is defined analogously. The precise way in which Vemuri proposes to estimate the coordinate functions  $f_j$  of the embedding is not important here – in principle, any one of a large number of surface reconstruction procedures (in the usual sense,  $z = f(x, y)$ ) could be used.

Hastie and Stuetzle [9] suggest a method for constructing a topological embedding of a planar region when the parameter vectors  $\lambda_i$  are unknown. In this case the  $\lambda_i$  as well as the embedding  $\mathbf{f}$  have to be estimated from the data. They propose an iterative algorithm, alternating between finding parameter vectors  $\lambda_i$  for a given embedding  $\mathbf{f}$ , and then finding  $\mathbf{f}$  for given  $\lambda_i$ . For given  $\mathbf{f}$ , the  $\lambda_i$  are found by projecting the data points  $\mathbf{x}_i$  onto the current estimate of the manifold:

$$\lambda_i = \operatorname{argmin}_{\lambda} \|\mathbf{x}_i - \mathbf{f}(\lambda)\|.$$

For given  $\lambda_i$ , the three coordinate functions of the embedding can be found individually by applying a function estimation procedure. The iteration is continued until it converges.

## References

- [1] J. D. Banfield and A. E. Raftery. Ice floe identification in satellite images using mathematical morphology and clustering about principal curves. Technical Report 172, Dept. of Statistics, U of Washington, 1989.
- [2] J.D. Banfield. *Constrained Cluster Analysis*. PhD thesis, Dept. of Statistics, U. of Washington, 1988.
- [3] R.M. Bolle and B.C. Vemuri. On three-dimensional surface reconstruction methods. *IEEE Trans. Pat. Anal. Mach. Intell.*, 13(1):1–13, January 1991.
- [4] Y. Breseler, J.A. Fessler, and A. Macovski. A Bayesian approach to reconstruction from incomplete projections of a multiple object 3d domain. *IEEE Trans. Pat. Anal. Mach. Intell.*, 11(8):840–858, August 1989.
- [5] J.F. Brinkley. Knowledge driven ultrasonic three-dimensional object modeling. *IEEE Trans. Pat. Anal. Mach. Intell.*, 7(4):431–441, July 1985.
- [6] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design*. Academic Press, second edition, 1990.
- [7] J.D. Foley. Interfaces for advanced computing. *Scientific American*, 257(4):126–135, October 1987.
- [8] T.A. Foley. Interpolation of scattered data over a spherical domain. In J.C. Mason and M.G. Cox, editors, *Algorithms for Approximation 2*, pages 303–310. Chapman and Hall, 1990.
- [9] T.J. Hastie and W. Stuetzle. Principal curves. *JASA*, 84:502–516, 1989.
- [10] J.F.P. Hudson. *Piecewise Linear Topology*. W.A. Benjamin, Inc, 1969.
- [11] P. Lancaster, , and K. Salkauskas. *Curve and Surface Fitting: An Introduction*. Academic Press, 1986.
- [12] W.S. Massey. *Algebraic Topology: An Introduction*. Harcourt, Brace & World, Inc., 1967.
- [13] J.W. Milnor. *Topology from the Differentiable Viewpoint*. University Press of Virginia, Charlottesville, Virginia, 1965.
- [14] D. Moore and J. Warren. Approximation of dense scattered data using algebraic surfaces. Technical Report Rice COMP TR90-135, Department of Computer Science, Rice University, 1990.

- [15] G.M. Nielson, T.A. Foley, B. Hamann, and D. Lane. Visualizing and modeling scattered multivariate data. *IEEE Computer Graphics and Applications*, 11(3):47–55, 1991.
- [16] M. Plass and M. Stone. Curve-fitting with piecewise parametric cubics. *Computer Graphics*, 17(3):229–239, July 1983.
- [17] V. Pratt. Direct least squares fitting of algebraic surfaces. *Computer Graphics*, 21(4):145–152, 1987.
- [18] P.J. Schneider. Phoenix: An interactive curve design system based on the automatic fitting of hand-sketched curves. Master’s thesis, Department of Computer Science, University of Washington, Seattle, WA., 1988.
- [19] R.B. Schudy and D.H. Ballard. Model detection of cardiac chambers in ultrasound images. Technical Report 12, Computer Science Department, University of Rochester, 1978.
- [20] R.B. Schudy and D.H. Ballard. Towards an anatomical model of heart motion as seen in 4-d cardiac ultrasound data. In *Proceedings of the 6th Conference on Computer Applications in Radiology and Computer-Aided Analysis of Radiological Images*, 1979.
- [21] B.I. Soroka. Generalized cones from serial sections. *Computer Graphics and Image Processing*, 15:154–166, 1981.
- [22] E.H. Spanier. *Algebraic Topology*. McGraw-Hill Inc., 1966.
- [23] G. Taubin. Estimation of planar curves, surfaces and nonplanar space curves defined by implicit equations, with applications to edge and range image segmentation. Technical Report LEMS-66, Division of Engineering, Brown University, 1990.
- [24] B.C. Vemuri. *Representation and Recognition of Objects From Dense Range Maps*. PhD thesis, Department of Electrical and Computer Engineering, University of Texas at Austin, 1987.
- [25] B.C. Vemuri, A. Mitiche, and J.K. Aggarwal. Curvature-based representation of objects from range data. *Image and Vision Computing*, 4(2):107–114, 1986.