Spline Smoothing on Surfaces

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http://www.stat.washington.edu/wxs/Surface-splines-01/thin-plate-6-14-02.pdf

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1. The function estimation problem

Given: Training data $(\underline{x}_1, y_1), \ldots, (\underline{x}_n, y_n)$ with $\underline{x}_i$ in some domain M and $y_i \in \mathbf{R}$.

Assumption: $y_i = f_{true}(\underline{x}_i) + \epsilon_i$.

Goal: Estimate f_{true}.

Well established method for $M = \mathbf{R}$: **spline smoothing**

Smoothing spline f_λ minimizes

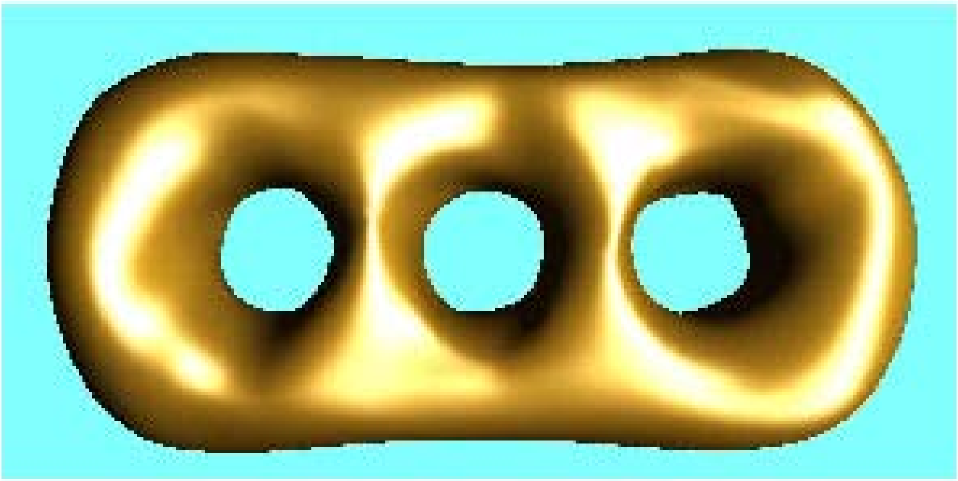
$$E[f] = \frac{1}{n} \sum (y_i - f(x_i))^2 + \lambda \int f''(x)^2 dx$$

in the Sobolev space of functions with square integrable second derivative.

Spline smoothing has previously been extended to the plane, the torus, and the sphere.

Objective:

Generalize spline smoothing to situations where domain M is a surface of complex topology and/or geometry.

Original motivation:

3D photography - modeling shape and appearance of real-world objects from sensor data.

(Un)fortunately, noise was low and smoothing proved to be unnecessary.

2. Smoothing on the line, revisited

Given: Training data $(x_1, y_1), \ldots, (x_n, y_n)$ with $a = x_1 < x_2 < \cdots < x_n = b$ and $y_i \in \mathbf{R}$

Goal: Find function f_λ minimizing *spline functional*

$$E[f] = \frac{1}{n} \sum (y_i - f(x_i))^2 + \lambda \int_a^b f''(x)^2 dx$$

in the Sobolev space $W_2([a, b])$ of functions with square integrable second derivative.

Will use this simple setting to introduce two (old) ideas:

- Approximation of f_λ using a finite element method;
- Evaluation of spline functions using subdivision.

Approximation of smoothing splines by finite elements

Suppose we have multi-resolution sequence of finite-dimensional function spaces

$$V^0 \subset V^1 \subset V^2 \subset \cdots \subset W_2([a,b])$$

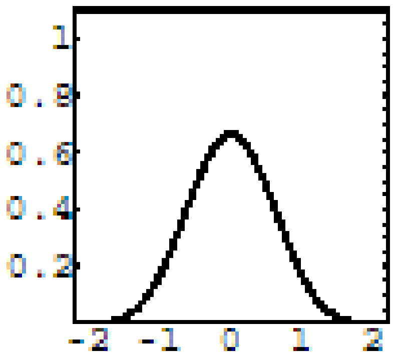
whose union is dense in $W_2([a,b])$.

Can then approximate f_λ by choosing resolution level J and minimizing $E[f]$ over V^J.

Example for multi-resolution sequence:

$V^J =$ restriction of space of cubic splines with knots on grid $\alpha\, 2^{-J}$ to $[a,b]$.

V^0 is spanned by translates ϕ_α^0 of a single basis function ϕ_0^0:



V^J is spanned by translated and scaled versions ϕ_α^J of ϕ_0^0:

$$\phi_\alpha^J(x) = \phi_0^0\left(2^J x - \alpha\right).$$

To find an approximate minimum for $E[f]$, choose a resolution level J and express $f(x)$ as a finite sum

$$f(x) = \sum_{\alpha} f_\alpha \phi_\alpha^J(x)\,,$$

where α ranges over the basis functions whose support intersects the interval $[a,b]$.

Substituting into the formula for the spline functional $E[f]$ gives

$$E[f] = \frac{1}{n}\sum_i \left(y_i - \sum_\alpha f_\alpha \phi_\alpha^J(x_i)\right)^2 + \lambda \sum_{\alpha,\beta} f_\alpha f_\beta B_{\alpha,\beta}\,,$$

with $B_{\alpha,\beta} = \int_a^b (\phi_\alpha^J)''(\phi_\beta^J)''dx$.

Determining the optimal coefficients f_α is a linear algebra problem.

Evaluation of splines by subdivision

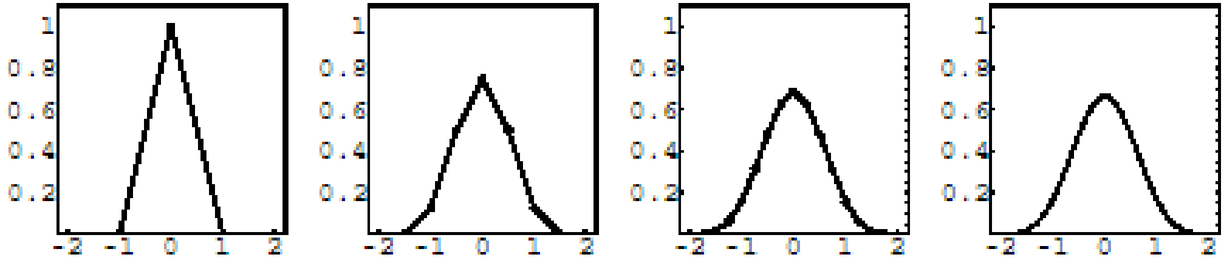
Let $f = \sum f_i \, \phi_i^0$ be a spline in V^0.

Let f_{PL}^0 be piecewise linear function on integer grid with $f_{PL}^0(i) = f_i$.

The function f_{PL}^0 can be regarded as a crude PL approximation to f.

Fact: We can obtain sequence of increasingly accurate PL approximations to the spline f by repeatedly

- Up-sampling the current PL approximation f_{PL}^J to a regular grid with half the spacing;
- Smoothing the values at the finer grid by a moving average with weights $1/4, 1/2, 1/4$.



2. Spline smoothing on the plane, revisited

Need:

- Replacement for ${f''}^2$;
- Multi-resolution sequence of spline spaces.

Replacements for ${f''}^2$:

- Squared Laplacian of f:

$$(tr(H_f))^2 = (\Delta f)^2 = (f_{x_1 x_1} + f_{x_2 x_2})^2\,;$$

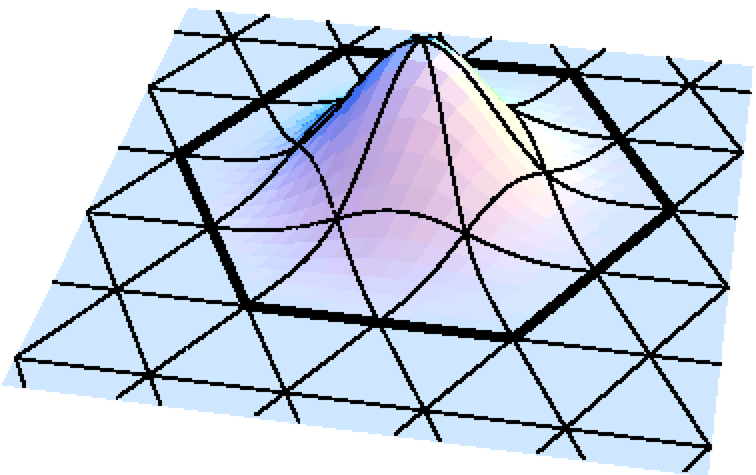
- Thin-plate energy:

$$tr(H_f^2) = (f_{x_1 x_1})^2 + 2\,(f_{x_1 x_2})^2 + (f_{x_2 x_2})^2\ .$$

Multi resolution sequence of spline spaces

Many options. Will consider quartic triangular B-splines.

V^0 is spanned by translates of a single basis function $\phi_0^0(\underline{x})$ to the vertices of a hexagonal lattice.



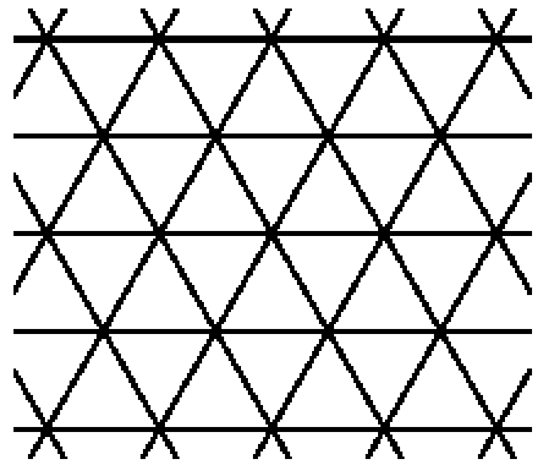


Figure 4. Hexagonal lattice with quartic triangular B-spline basis function ϕ_0^0. The support of the basis function consists to the 24 triangles enclosed by the solid hexagon.

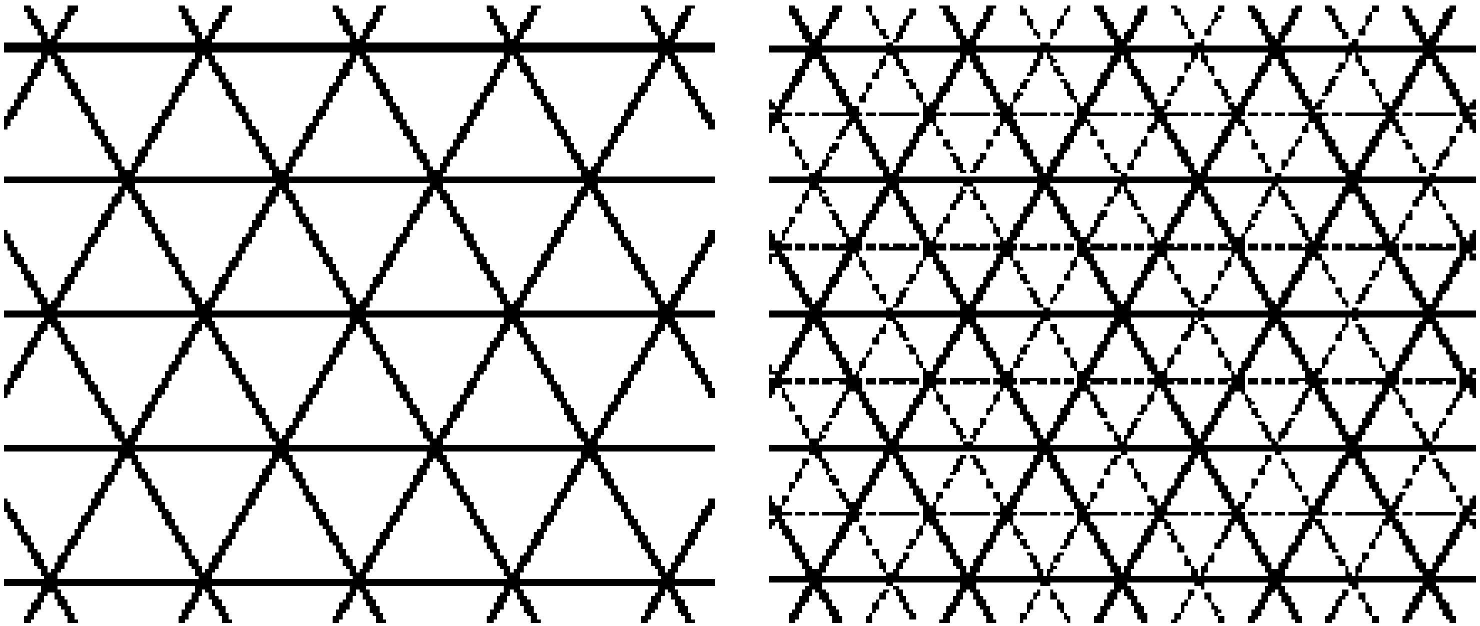
Basis function ϕ_0^0 is unique function with following properties:

- It is quartic polynomial on each triangle;
- It is C^2;
- It has minimum support;
- Its translates form a partition of unity.

The space V^J at resolution J is spanned by scaled and translated versions $\phi_{\underline{a}}^J$ of ϕ_0^0:

$$\phi_{\underline{a}}^J(\underline{x}) = \phi_0^0(2^J x_1 - a_1, 2^J x_2 - \underline{a}_2)\,.$$

Basis functions of V^1 are centered at vertices of 4-1 subdivided lattice for V^0.



Evaluation of quartic triangular B-splines by subdivision

Let $f = \sum f_{\underline{a}} \phi^0_{\underline{a}}$ be a spline in V^0.

Let f^0_{PL} be the piecewise linear function whose value at the center vertex of $\phi^0_{\underline{a}}$ is $f_{\underline{a}}$.

Fact: We can obtain a sequence of increasingly accurate PL approximations to the spline f by repeatedly

- Up-sampling the current PL approximation f^J_{PL} to the vertices of a 4-1 subdivided version of the current lattice;
- Smoothing the values at the vertices of the subdivided lattice.

Smoothing:

Let v_0 be a vertex of the subdivided lattice, and let $v_1, \ldots, v_n$ be its neighbors. Then

$$f_{PL}^{J+1}(v_0) = \frac{1}{4} f_{PL}^{J}(v_0) + \frac{1}{8} \sum_i f_{PL}^{J}(v_i) \, .$$

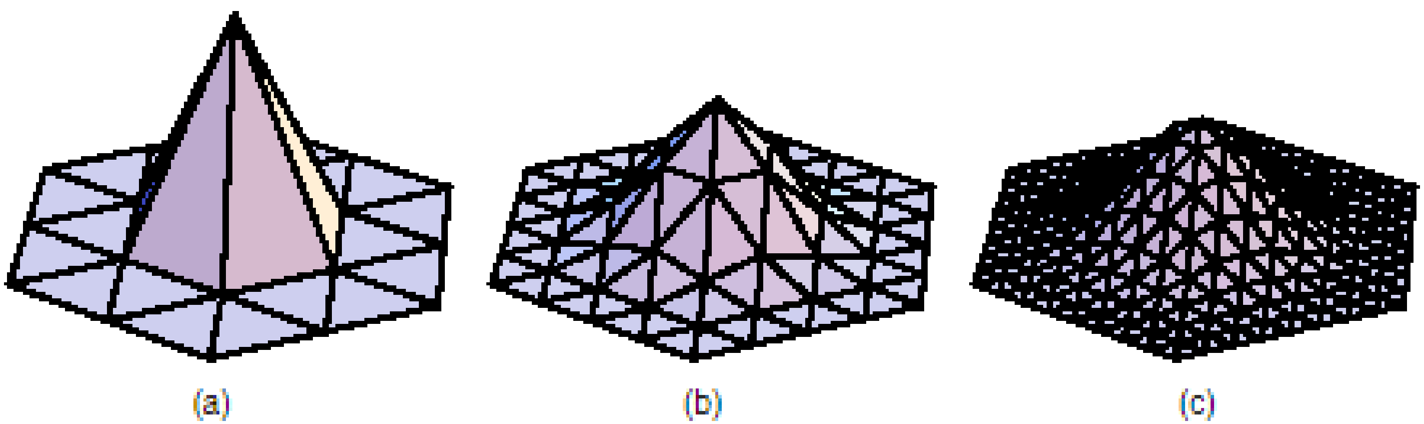


Figure 5. The subdivision process applies to the hat function $\hat{\phi}_0^0$ (left) yields a sequence of piecewise linear functions that converges to the quartic triangular B-spline basis function ϕ_0^0 (right).

3. Abstract spline smoothing on smooth surfaces

Let $M \subset \mathbf{R}^3$ be smooth surface and let f be function on M.

To generalize spline smoothing, need measure for roughness of f.

Consider point $p \in M$. Wlog assume that that p is the origin and the tangent plane at p is the (x_1, x_2)-plane.

Near p the surface has a local parameterization

$$M = \{\underline{x} \,|\, \underline{x} = (x_1, x_2, F(x_1, x_2))\}.$$

Any function on M can (locally) be regarded as a function $f(x_1, x_2)$.

Hessian of f at $p =$ matrix of second derivatives of f at the origin.

Value of *Laplace-Beltrami* operator Δ_M at $p =$ trace of the Hessian.

Can now formally define spline smoothing problem on M:

Goal: Find function f_λ minimizing spline functional

$$E[f] = \frac{1}{n}\sum(y_i - f(x_i))^2 + \lambda\int_M(\Delta_M f)^2 dA$$

in the Sobolev space $W_2(M)$ of functions with square integrable second derivative.

No closed form solution except in special cases (line, sphere, torus).

To use finite elements, need

- Multi-resolution sequence of finite dimensional subspaces of $W_2(M)$;
- Way of evaluating $\Delta_M f$.

4. Operational spline smoothing on surfaces

Suppose we had a polyhedron K with triangular faces that was homeomorphic to M. (Not a restriction - every surface can be triangulated.)

Subdivision functions on K are a natural generalization of quartic triangular B-splines.

To define a resolution level 0 subdivision function, start with a function f_{PL}^{0} that is piecewise linear on $K^0 = K$, with values f_α at the vertices.

The function f_{PL}^{J} is piecewise linear on K^J (obtained by J 4-1 splits of K^0).

The values of f_{PL}^{J+1} at the vertices of K^{J+1} are obtained by

- Up-sampling f_{PL}^{J} to the vertices of K^{J+1};
- Local averaging.

The subdivision function defined by f^0_{PL} is the limit of this process.

The resolution level 0 subdivision functions form a vector space V^0 with dimension = number of vertices of K.

Resolution level J subdivision functions are obtained by fixing the values of a piecewise linear function at the vertices of K^J and running the subdivision process.

Choice of weights for averaging is critical.

Rule for degree 6 vertices is the same as in the case of quartic triangular B-splines $\Rightarrow$ subdivision functions are quartic triangular B-splines except at *extraordinary* vertices.

Rules for *extraordinary vertices* have been carefully crafted to make subdivision functions "smooth": If we embed K into R^3 using subdivision functions, the resulting surface is smooth (essentially C^2).

We assume that surface M is a subdivision surface (an embedding of a polyhedron whose coordinate functions are subdivision functions).

Then the function spaces V^J are in $W_2(M)$ and their union is dense in $W_2(M)$.

Note:

Any smooth surface can be approximated by a subdivision surface.

There are algorithms to construct such approximations.

Review:

We are given a subdivision surface M and data $(\underline{x}_1, y_1), \ldots, (\underline{x}_n, y_n)$ with $\underline{x}_i \in M$ and $y_i \in \mathbf{R}$.

We want to approximate the function f_λ minimizing the *spline functional*

$$E[f] = \frac{1}{n} \sum (y_i - f(x_i))^2 + \lambda \int_M (\Delta_M f)^2 dA$$

in the Sobolev space $W_2(M)$ of functions with square integrable second derivative.

We assume that M is a subdivision surface $\Rightarrow$

We have a multiresolution sequence of subdivision function spaces $V^0 \subset V^1 \subset \cdots \subset W_2(M)$

To find an approximate minimum for $E[f]$, choose a resolution level J and express $f(x)$ as a finite sum

$$f(x) = \sum_{\alpha} f_\alpha \phi_\alpha^J(x)\,,$$

where α ranges over the basis functions.

Substituting into the formula for the spline functional $E[f]$ gives

$$E[f] = \frac{1}{n}\sum_i \left(y_i - \sum_\alpha f_\alpha \phi_\alpha^J(x_i)\right)^2 + \lambda \sum_{\alpha,\beta} f_\alpha f_\beta B_{\alpha,\beta}\,,$$

with

$$B_{\alpha,\beta} = \int_M \Delta_M \phi_\alpha^J \, \Delta_M \phi_\beta^J \, dA\,.$$

We solve the resulting linear algebra problem using preconditioned conjugate gradients.

5. Concluding remarks:

In the paper we compare our method to the standard approach in the case of the sphere where the splines can be computed exactly.

Error decreases exponentially with subdivision level and is decreasing in λ (not surprising).

We also have an example suggesting that generalized cross-validation is a viable way of choosing the smoothing parameter λ.

Our code only handles surfaces without boundary, but the ideas generalize.

Instead of choosing a fixed resolution level, could use wavelet methods.